Indian J.Sci.Res. 5(2) : 181-189, 2014 Research Paper ISSN : 0976-2876 (Print)
ISSN : 2250-0138 (Online)

A NOTE ON FRACTIONAL CALCULUS FOR GENERALIZED FUNCTIONS

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ABSTRACT

We have to study of the fractional calculus relative to the certain spaces of testing functions and corresponding spaces of generalized functions. There is a lot of work has been carried out on the concept of fractional calculus in the framework of classical functions but as for as the study of the same concept concerned generalized functions is going on and like to mention few such asthe Riemann- Liouville fractional integral, the Weyl fractional integral and their generalizations Erdelyi-Kober fractional integrals on (0,∞) have been extended to generalized functions by Erdelyi(1972) and McBride(1979). On the other hand Jones(1970-72) has extended the operators H_α and $K_\alpha$within the framework of his generalized functions and Pathak (1990) has extended the same operators to certain Schwartz distributions. The aim of this paper is to define the fractional integrals and the fractional derivatives on the generalized function space D' following (1990).

KEYWORDS: Fractional Calculus, Riemann- Liouville Fractional Integral, The Weyl Fractional Integral, Test Functions and Generalized Functions

Fractional calculus is the study of the derivatives and integrals of arbitrary order (real or complex). This concept is not new, it is as old as the ordinary calculus. Several mathematicians contributed to this subject over the years. But in the 20th century notable contributions have been made to both the theory and application of the fractional calculus Authors like Liouville, Riemann (1876), and Weylmade major contributions to the theory of fractional calculus. Theory of fractional calculus for the classical functions is now well known and it is systematically available in various standard texts such as (Oldham and Spainer, 1974), (Samko *et al.*, 1993). Also along with the development of the theory of the generalized functions has been continued with contributions from Zemanian (1968), Gel'fand and Shilov(1967), McBride(1970-72), Erdelyie (1972), J.N. Pandey (1983), R.S. Pathak (1990, 1994)and others.

PRELIMINARIES

Here we mainly discuss, in brief, few more interesting studies of fractional calculus along with used fractional operators, especially Riemann- Liouville fractional operator & The Weyl fractional operator and their certain generalizations in the framework of classical functions. Then we discuss some approaches of extension of fractional operators from classical functions to generalized functions through the introduction of McBride spaces & Schwartz spaces.

In 1890, first S.F Lacroix (1819) has generalized the formula

$$D^n z^m = \frac{n!}{(m-n)} z^{m-n}$$

where m and n are natural numbers, for derivatives of arbitrary order α as

$$D^\alpha z^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} z^{m-\alpha} \quad (1.1)$$

where the only restriction is that m≠-1,-2,….. Then he formally replace α with the fraction ½ and together with the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, He obtained,

$$D^{1/2} z = \frac{d^{1/2} z}{dz} = \frac{2\sqrt{z}}{\sqrt{\pi}}$$

We can now define the fractional derivative of $z^m f(z)$, where *f(z)* is analytic at z=0, by differentiating the power series for $z^m f(z)$ term by term. We get

$$D^\alpha Z^m f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(m+n+1) f^{(n)}(0) z^{m-\alpha+n}}{\Gamma(m-\alpha+n+1)} \quad (1.2)$$

The series has the same circle of convergence as the power series for *f(z)* about z=0

Following Oldham (1974), The definition of fractional derivative given by Grunwald (1867) and later extended by Post (1930) is considered as most fundamental because it involved the fewest restrictions on the functions to which it applies and avoids explicit use of the notions of arbitrary derivative and integral. And it defined the derivative of arbitrary order α by the formula,

$$D^\alpha f(x) = \underset{N\to\infty}{Lim} \left\{ \frac{\left(\frac{x-\alpha}{N}\right)^{-\alpha}}{\Gamma(-\alpha)} \sum_{m=0}^{N-1} \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} f\left(x - m\left[\frac{x-\alpha)}{N}\right]\right) \right\} \quad (1.3)$$

Note: For α as a non-negative integer Γ(-α) is infinite but the ratio Γ(m-α)/Γ(-α) is finite

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The most usual approach for generalization of ordinary derivative and integral to arbitrary order is known as Riemann-Liouville integral(s).

The integral

$$(I^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt \quad 0<x<\infty \quad (1.4)$$

which defined fractional integration, is called Riemann-Liouville fractional integral of order α for Re(α) > 0 and for suitable functions *f*. This integral can be motivated from the cauchy formula for a repeated integral.

$$\int_0^x dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)dt_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt$$

for n=1,2,… and $0<x<\infty$. Now it can be easily generalized to non integer values and gives (1.4).

the adjoint of the integral I^{α} is an operator K^{α} defined by

$$(K^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{\alpha-1}f(t)\,dt \quad 0<x<\infty \quad (1.5)$$

for Re(α) > 0. It also defined fractional integral is called *Weyl fractional integral* of order α.

Fractional integrals (1.4) and (1.5) are defined for functions $f(x) \in L_1(0,\infty)$, existing almost everywhere.

We now define fractional derivative of a *locallyintegable functionf* by means of the relation

$$(I^{-\alpha} f)(x) = D^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x (x-t)^{-\alpha-1} f(t)\,dt \quad (1.6)$$

which is know as Riemann-Liouville fractional derivative of order α for Re(α) < 0.

Similarly the Weyl fractional derivative is defined by

$$(K^{-\alpha} f)(x) = \frac{1}{\Gamma(-\alpha)}\int_x^{\infty} (t-x)^{\alpha-1} f(t)\,dt \quad \text{for } Re(\alpha) < \quad (1.7)$$

Remarks

(i) The definitions of fractional derivative given by (1.4) and (1.6) can be extended to $Re(\alpha) \ge 0$ as follow.

$$(I^{-\alpha} f)(x) = \frac{d^n}{dx^n}(I^{-(\alpha-n)} f)(x) \quad (1.8)$$

and $$(K^{-\alpha} f)(x) = (-1)^n \frac{d^n}{dx^n}(K^{-(\alpha-n)} f)(x) \quad (1.9)$$

where n=(*Re(α))+1*

(ii) If the lower integration limit in each of (1.4) and (1.6) is c then the operators $I^{\alpha}$and $I^{-\alpha}$aredenoted by ${}_C I_x^{\alpha}$ and ${}_C I_x^{-\alpha}$ respectively. Similarly the operators $K^{\alpha}$and $K^{-\alpha}$are replaced by ${}_b K_x^{\alpha}$ and ${}_b K_x^{-\alpha}$ if the upper limit in each of (1.5) and (1.7) is b<∞. Therefor if $f(x) \in L_1$(a,b) the Riemann-Liouville and Weyl fractional integral operators are defined, for Re(α)=0, by

$$({}_a I^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \ x>a \quad (1.10)$$

and

$$({}_b K^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}dt, \ x<a \quad (1.11)$$

The operators given by (1.10) and (1.11) can be easily extended from the case of finite interval (a,b) to the case of half axis, given by

$$(I^{\alpha} f)(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \quad 0<x<\infty \quad (1.12)$$

and to the whole (or entire) real axis, by

$$(I^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1} f(t)dt, \quad -\infty<x<\infty \quad (1.13)$$

$$(K^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{\alpha-1} f(t)\,dt, \quad -\infty<x<\infty \quad (1.14)$$

It is known that in the case of finite interval the operators given by (1.10) and (1.10) are defined on any space L_P,1 ≤ p ≤∞ and they have mapped L_P, 1< p <∞, into L_q with q such that 1 ≤ q ≤ p/(1-αp) when αp < 1 and 1 ≤ q <∞ when αp ≥ 1. But for the case of the whole axis or half axis these operators are well defined on the space L_P,1 ≤ p ≤ $1/\alpha$ and they may map L_p into L_q for,1 ≤ p ≤ $1/\alpha$and q=p(1-αp) only.

The fractional integral operators I^{α} and K^{α} defined by (1.13) and (1.14) have been studied by Erdelyi and Kober (1940). Their application can be found in the works of Sneddon (1966), Noble and Whiteman (1970).

The next method for motivating the concept of derivative of arbitrary order stems from consideration of Cauchy's integral formula

$$D^n f(z) = \frac{n!}{2\pi i} \oint_c \frac{f(w)}{(w-z)^{n+1}} dz \qquad (1.15)$$

where c describes a closed contour surrounding the point z and enclosing a region of analyticity of f. When the positive integer n is replaced by a non-integer α, then $(w-z)^{-\alpha-1}$no longer has a pole at $w=z$ but a branch point. One is no longer free to deform the contour $c$surrounding z at will, Since the integral will depend on the location of the point at which c crosses the branch line for $(w-z)^{-\alpha-1}$. This point is chosen to be 0 and the branch line is to be the straight line joining 0 and z and continuing indefinitely in the quadaent$Re(w) \leq 0$, $I_m(w) \leq 0$. Then one simple defines, for α not a negative integer,

$$D^\alpha f = \frac{\Gamma(\alpha+1)}{2\pi i} \oint_c \frac{f(w)}{(w-z)^{\alpha+1}} dw \qquad (1.16)$$

where the contour c begins and ends at $w=0$ enclosing z once in the positive sense.

The definition (1.15) is attributed by Osler (1970) to Nekrassov (1888).

It can be proved that this generalization of the ordinary derivative is equivalent to the Riemann-Liouville derivative for the appropriate value of α in which both derivative are defined.

These approaches and others are discussed and compared by Osler (1970), by Ross (1966) and by many other authors.

Generalizations of Riemann-Liouville and Weyl Fractional Integral operators

We now discuss briefly a generalization of Riemann-Liouville and Weyl fractional operators.

The operators (1.4) and (1.5) can be generalized in two ways.

Firstly, we may wish to integrate with respect to a continuously differentiable function ρ of a positive real variable, producing an expression such as-

$$\frac{1}{\Gamma(\alpha)} \int_0^x (\rho(x)-\rho(t))^{\alpha-1} f(t) \rho'(t) dt$$

For ρ (x) = x^σ (σ real and σ> 0), the operator I_σ^α and K_σ^α defined for Re (α) >0 and suitable function f by

$$(I_\sigma^\alpha f)(x) = \frac{\sigma}{\Gamma(\alpha)} \int_0^x (x^\sigma - t^\sigma)^{\alpha-1} t^{\sigma-1} f(t) dt \qquad (2.1)$$

$$(K_\sigma^\alpha f)(x) = \frac{\sigma}{\Gamma(\alpha)} \int_x^0 (t^\sigma - x^\sigma)^{\alpha-1} t^{\sigma-1} f(t) dt \qquad (2.2)$$

The case σ = 1, of course, takes us back to (1.4) and (1.5) again.

Secondly, in a series of papers (2,3,4), A. Erdelyi and H. Kober investigated the properties of the following generalization of Riemann-Liouville and Weyl fractional integrals:

$$(I^{n,\alpha} f)(x) = \frac{x^{-\eta-\alpha-1}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{n-1} f(t) dt \qquad (\alpha>0,\ \eta>0)$$

(2.3)

$$(K^{n,\alpha} f)(x) \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-n-\alpha} f(t) dt \qquad (\alpha>0,\ \eta>0)$$

(2.4)

These operators, if generalized, on the pattern of (2.1) and (2.2) can be put into the form

$$(I_{x^\sigma}^{\eta,\alpha} f)(x) = \frac{\sigma x^{-\sigma\eta-\sigma\alpha}}{\Gamma(\alpha)} \int_0^x (x^\sigma - t^\sigma)^{\alpha-1} t^{\sigma\eta+\sigma-1} f(t) dt \qquad (2.5)$$

$$(K_{x^\sigma}^{\eta,\alpha} f)(x) = \frac{\sigma x^{-\sigma\eta}}{\Gamma(\alpha)} \int_x^\infty (t^\sigma - x^\sigma)^{\alpha-1} t^{-\sigma\eta-\sigma\alpha+\sigma-1} f(t) dt$$

(2.6)

The case σ = 2 i.e.

$$(I_{x^2}^{\eta,\alpha} f)(x) = I(\eta,\alpha) f(x) = \frac{2x^{-2\eta-2\alpha}}{\Gamma(\alpha)} \int_0^x (x^2 - t^2)^{\alpha-1} t^{2\eta+1} f(t) dt$$

(2.7)

$$(K_{x^2}^{\eta,\alpha} f)(x) = K(\eta,\alpha) f(x) = \frac{2x^{-2\eta}}{\Gamma(\alpha)} \int_x^\infty (t^2 - x^2)^{\alpha-1} t^{-2\eta-2\alpha+1} f(t) dt$$

(2.8)

has been extensively studied by I.N. Sneddon (1966) and he has been obtained the relations between (2.7), (2.8) and the modified operators of Hankel transform

$$S_{\eta,\alpha} f(x) = 2^\alpha x^{-\alpha} \int_0^\infty t^{1-\alpha} J_{2\eta+\alpha}(xt) f(t) dt$$

(2.9)

and then applied them to obtain solutions of dual, triple and quadruple integral equations.

L.G. Makarenio has introduced the two dimensional form of the operators (1.4) and (1.5) as

$$(Ff)(x,y) = F(\xi,\alpha,\eta,\beta:\sigma,\delta)\, f(x,y) = \frac{\sigma\delta\, x^{-\sigma(\xi+\alpha-1)}\, y^{-\delta(\eta+\beta-1)}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x\int_0^y u^{\sigma\xi-1} v^{\sigma\eta-1}$$

$$\times (x^\sigma - u^\sigma)^{\alpha-1} (y^\sigma - v^\sigma)^{\beta-1} f(u,v)\, du\, dv \quad (2.10)$$

$$(Kf)(x,y) = K(\xi,\alpha,\eta,\beta:\sigma,\delta)\, f(x,y) = \frac{\sigma\delta\, x^{\sigma\xi}\, y^{\delta\eta}}{\Gamma(\alpha)\Gamma(\beta)} \int_x^\infty\int_y^\infty u^{-\sigma(\xi+\alpha-1)-1} v^{-\delta(\eta+\beta-1)-1}$$

$$\times (u^\sigma - x^\sigma)^{\alpha-1} (v^\sigma - y^\sigma)^{\beta-1} f(u,v)\, du\, dv \quad (2.11)$$

The operators (2.5) and (2.6) have been further generalized by Roonie (1978) as follows :

For m=0,1,2,… and Re $\alpha > 0$, let

$$k_{m,\alpha}(x) = \begin{cases} -\sum_{r=0}^{m-1} \dfrac{(-1)^r}{\Gamma(r+1)\,\Gamma(\alpha-r) x^r}, & 0 < x < 1 \\ \dfrac{(1-x^{-1})^{\alpha-1}}{\Gamma(\alpha)} - \sum_{r=0}^{m-1} \dfrac{(-1)^r}{\Gamma(r+1)\Gamma(\alpha-r) x^r}, & x > 1 \end{cases} \quad (2.12)$$

where the empty sum which occurs if m=0 is defined to be zero; and also let $l_{m,\alpha}(x) = k_{m,\alpha}(x^{-1})$. If $\sigma>0$ and ξ, η are complex numbers, defined

$$l_{m,\sigma,\alpha,\xi}\, f(x) = \sigma \int_0^\infty (x/t)^{-\sigma\xi}\, k_{m,\alpha}((x/t)^\alpha)\, f(t) \frac{dt}{t} \quad (2.13)$$

$$k_{m,\sigma,\beta,\eta}\, f(x) = \sigma \int_0^\infty (x/t)^{\sigma\eta}\, l_{m,\beta}((x/t)^\sigma)\, f(t) \frac{dt}{t} \quad (2.14)$$

It is easily shown that

$$I_{0,\sigma,\alpha\xi} = I_{x^\sigma}^{\alpha,\xi} \quad \text{and} \quad K_{0,\sigma,\beta,\eta} = K_{x^\sigma}^{\eta,\beta} \quad (2.15)$$

Lowndes (1970) has introduced the generalized Erdelyi-Kober operators

$$I_k(\eta,\alpha) f(x) = 2^\alpha k^{1-\alpha} x^{-2\eta-2\alpha} \int_0^x u^{2\eta+1} (x^2 - u^2)^{1/2(\alpha-1)} J_{\alpha-1}\{k\sqrt{x^2-u^2}\} f(u)\, du \quad (2.16)$$

$$K_k(\eta,\alpha) f(x) = 2^\alpha k^{1-\alpha} x^{2\eta} \int_x^\infty u^{-2\eta-2\alpha+1} (u^2 - x^2)^{1/2(\alpha-1)} J_{\alpha-1}\{k\sqrt{u^2-x^2}\} f(u)\, du \quad (2.17)$$

and the generalized operators of Hankel transform

$$S\begin{pmatrix} a,b,k \\ \eta,\alpha,\sigma \end{pmatrix} f(x) = 2^\alpha x^{2\sigma-\alpha} (x^2 - a^2)^{-\sigma} \int_k^\infty u^{1-2\sigma-\alpha} (u^2 - k^2)^\sigma J_{2\eta+\alpha}\{\sqrt{(x^2-a^2)(u^2-b^2)}\} f(u)\, du \quad (2.18)$$

and obtained some relations between them, which are then applied to solve a pair of dual integral equations.

Heywood and Rooney (1975) have written the Lowndes operators (2.16) and (2.17) in the form

$$I_k(\eta,\alpha) f(x) = 2^{\alpha-1} k^{1-\alpha} x^{-\eta-\alpha} \int_0^x u^\eta (x - u)^{1/2(\alpha-1)} J_{\alpha-1}\{k\sqrt{x-u}\} f(u)\, du \quad (2.19)$$

$$K_k(\eta,\alpha) f(x) = 2^{\alpha-1} k^{1-\alpha} x^\eta \int_x^\infty u^{-\eta-\alpha} (u - x)^{1/2(\alpha-1)} J_{\alpha-1}\{k\sqrt{u-x}\} f(u)\, du \quad (2.20)$$

They have expressed the operator $I_k(\eta,\alpha)$ in terms of an auxiliary operators $R_{k,\nu,0}$ by the equation

$$I_k(\eta,\alpha) f(x) = x^{-\eta} I_k(o,\alpha)[t^n\, f(t)](x), \quad (2.21)$$

where

$$I_k(o,\alpha) f(x) = R_{k,\nu,\alpha} f(x) = x^{-\alpha} (H_{\nu+\alpha} T_k H_\nu f)(x), \quad (2.22)$$

$$H_\nu f(x) = x^{(1/2)\nu} \int_0^\infty t^{-(1/2)\nu} J_\nu(2\sqrt{xt}) f(t)\, dt \quad (2.23)$$

and T_k is translation operator defined by

$$(T_k f)(x) = \begin{cases} f\left(x - \frac{1}{4}k^2\right), & x > \frac{1}{4}k^2 \\ 0, & 0 < x \le \frac{1}{4}k^2 \end{cases} \quad (2.24)$$

The operators $K_k(\eta, \alpha)$ is expressed by Heywood as

$$K_k(\eta,\alpha) f(x) = x^\eta (K_k(o,\alpha)[t^{-\eta} f(t)])(x), \quad (2.25)$$

where

$$K_k(o,\alpha) f(x) = (S_{k,\nu,\alpha} f)(x) \quad (2.26)$$

THE OKIKIOLU AND RIESZ FRACTIONAL INTEGRAL OPERATORS

The fractional integral operator H_α defined by

$$(H_\alpha f(x) = \frac{1}{\pi}\Gamma(1-\alpha)\cos{}^{1\!}/_{2}\,\pi\alpha\int_{-\infty}^{\infty} f(t)\frac{\operatorname{sgn}(t-x)}{|t-x|^{1-\alpha}}dt \tag{3.1}$$

was introduced by Okikiolu It reduces to Hilbert transform for $\alpha=0$. When $0<\alpha<1$, the above integral is absolutely convergent for a suitably restricted f(t).

A variant of H_α is the Riesz fractional integral operator K_α defined for $0<\alpha<1$, by

$$(K_\alpha f(x) = \frac{1}{\pi}\Gamma(1-\alpha)\,Sin\,{}^{1\!}/_{2}\,\pi\alpha\int_{-\infty}^{\infty} f(t)\frac{f(t)}{|t-x|^{1-\alpha}}dt \tag{3.2}$$

Also the inversion formula for (3.1) and (3.2) are defined as;

If $f\in L^P(-\infty,\infty)$ where $1<P<\alpha^{-1}$ and if $g(x)=(H_\alpha f)(x)$ then

$$f(x)=(H_\alpha^{-1}g)(x)=-(H_{-\alpha}g)(x)=\lim_{\delta\to 0+}\frac{1}{\pi}\Gamma(1+\alpha)\cos(\frac{1}{2}\pi\alpha)\,T_\delta(g)(x) \tag{3.3}$$

where $$T_\delta(g)(x)=\int_{|t-x|\ge\delta}\frac{\operatorname{sgn}(t-x)}{|(t-x)|^{1+\alpha}}g(t)dt$$

point-wise almost every where and also in the p-norm.

And if $g\ (x)=(K_\alpha f)(x)$ then

$$f(x)=(K_\alpha^{-1}g_1)(x)=\lim_{\delta\to 0+}\frac{1}{\pi}\Gamma(1+\alpha)\sin(\frac{1}{2}\pi\alpha)$$
$$\times\int_{|(t-x)|\ge\delta}\frac{g_1(t)-g_1(x)\,|}{|(t-x)|^{1+\alpha}}dt \tag{3.4}$$

point-wise almost every where and also in the L^P- norm.

Fractional Derivative with Respect to An Arbitrary Function

The concept of fractional derivative with respect to a function has been introduced by Erdelyi (1972), (1940). This concept is very useful and suggestive in applications Erdelyi first defined αth-order *differintegral* of function *f(z)* with respect to the function z^n by the formula

$${}_aD_{Z^n}^{\alpha}\ f(z)=\frac{1}{\Gamma(-\alpha)}\int_a^z\frac{f(w)w^{n-1}}{(z^n-w^n)^{1+\alpha}}dw \tag{4.1.1}$$

Osler (1970) has extended Erdelyi's work by defining a differintegral of a function *f(z)* with respect to an arbitrary function *g(z)* by considering the Riemann-Liouville integral, as

$${}_aD_{g(z)}^{\alpha}f(z)=\frac{1}{\Gamma(-\alpha)}\int_a^z\frac{f(w)g'(w)}{[g(z)-g(w)]^{1+\alpha}}dw \tag{4.1.2}$$

where a is chosen to give *g(a)=0*, i.e. $a=g^{-1}(0)$.

If we take *g(w)= u g(z)*, we get

$$D_{g(z)}^{\alpha}f(z)=\frac{g(z)^{-\alpha}}{\Gamma(-\alpha)}\int_0^1\frac{f(g^{-1}(ug(z)))}{(1-u)^{\alpha+1}}du \tag{4.1.3}$$

In particular, if we set *g(z)= z-a*, then (5.2.12) reduces to the Riemann-Liouville integral

$$D_{z-a}^{\alpha}f(z)=\frac{1}{\Gamma(-\alpha)}\int_a^z\frac{f(w)}{(z-w)^{\alpha+1}}dw \tag{4.1.4}$$

It is noted that certain choices of g have been shown by Erdelyi and by Osler to lead to a number of formula of interest in classical analysis.

Fractional Partial Derivative

Riesz (1949) and Bassan (1961) introduced the concept of fractional partial derivatives. The notation $D_{g(z),h(\omega)}^{\alpha,\beta}f(z,w)$ means the fractional derivative of *f(z,w)* of order β with respect to h(*w*) holding z fixed, followed by the derivative of order α with respect to g(z) holding *w* fixed. This is defined by

$$D_{g(z),h(w)}^{\alpha,\beta}f(z,w)=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{-4\pi^2}\int_{g^{-1}(0)}^{z}\frac{g'(\xi)}{[g(\xi)-g(z)]^{\alpha+1}}$$
$$\times\int_{h^{-1}(0)}^{\omega}\frac{f(\xi-\zeta)h'(\zeta)}{[h(\zeta)-h(w)]^{\beta+1}}d\zeta d\xi \tag{4.2.1}$$

where*f(z,w)*, *g(z)* and *h(w)* are assumed to possess sufficient regularity to give the definition meaning.

When *f(z,w)= u(z) v(z)*, then from (1.15) we have

$$D^{\alpha,\beta}_{g(z),g(w)}u(z)v(w)/w = D^{\alpha,\beta}_{g(z)}u(z)\,D^{\beta}_{g(z)}v(z)$$
(4.2.2)

Fractional Integrals of Generalized Functions

The extension of fractional calculus from classical functions to generalized functions is getting more attention in present time. Mention must be made of the works of Zemanian (1968), Gel'fand and Shilov (1967), Erdelyie (1972), Pandey and Chaudhary (1983), Pathak (1990), Pathak and Upadhyay, (1994)and others.

It is very important to note that in such an extension of the fractional operators from classical functions to generalized functions, it is needed to create space of testing functions and corresponding space of generalized functions. In accordance, we describe below certain spaces of testing functions and corresponding spaces of generalized functions such as Schwartz spaces, McBride spaces.

Schwartz's Spaces $\mathbf{D}_L^p$ and $(\mathbf{D}_L^p)'$

For $1 \le P \le \infty$ $\mathbf{D}_L^p$ defined by Schwartz (83) is given by

$\mathbf{D}_L^p$ (R) = $\{\phi : \phi \in C^\infty \;\; and \;\; D_x{}^k \phi \in L^p \;\; (k = 0,1,2,......)\}$
(5.1.1)

The topology over $\mathbf{D}_L^P$ is generated by the following seminorms

$$\gamma_k{}^P(\phi) = \left\| \frac{d^k}{dx^k}\phi \right\|_P \qquad ,k = 0,1,2,....$$
(5.1.2)

$\mathbf{D}_L^p$ is a complete countablymultinormed space and hence a Frechet space. A sequence $\{\phi_v\} \in \mathbf{D}_L^p$ converges to zero in $\mathbf{D}_L^p$ if $\phi_v^{(k)}$ converges to zero in L^P for each $k \in N_0,\ as\ v \to \infty$.

Following Schwartz (1950-51), we denote the dual of $\mathbf{D}_L^p$, $1 < P \le \infty$ by $(\mathbf{D}_L^p)'$ or $\mathbf{D}_L^q$, where

$$\frac{1}{P} + \frac{1}{q} = 1.$$

D is dense in $\mathbf{D}_L^p$ ($1 < P < \infty$) and convergence in D implies convergence in $\mathbf{D}_L^p$ and consequently by the restriction of $f \in (\mathbf{D}_L^p)'$ to D is in D′. it can be easily seen that

$$\varepsilon' \subset (\mathcal{D}_L{}^p)' \subset D'$$

Schwartz also defined the space $\dot{B}$ as

$$\dot{B} = \left\{ \phi \in C^\infty(-\infty,\infty) : \frac{d^k}{dx^k}\phi \to 0 \ as \ |x \to \infty| \quad for\ k = 0,1,2,..... \right\}$$
(5.1.3)

The space $\dot{B}$ is equipped with the topology generalized by the seminorms

$$\gamma_k{}^\infty(\phi) = \left\| \frac{d^k}{dx^k}\phi \right\|_{L^\infty(-\infty,\infty)}$$
(5.1.4)

It is obvious that the Schwartz space $\dot{B}$ is a subspace of $\mathbf{D}_L^\infty$ consisting of all functions which vanish at infinity together with each of their derivatives. The dual of $\dot{B}$ is the space $(\mathbf{D}_L^1)'$.

Further Pathak in 1990 has extensively studied the Riesz fractional operator K_α and Okikialu fractional integral operator H_α on the spaces $\mathbf{D}_L^P$ and $(\mathbf{D}_L^P)'$.

McBride Spaces, $F_{P,\mu}$ and $F'_{P,\mu}$:

McBride (1975-76) introduced the spaces $F_{p,\mu}$of testing functions and the corresponding spaces $F'_{p,\mu}$of generalized functions as follows:

For $1 \le P < \infty$,

$$F_P = \left\{ \phi \in C^\infty(0,\infty) : x^k \frac{d^k}{dx^k}\phi \in L^P(0,\infty) \qquad k = 0,1,2,..... \right\}$$
(5.1.5)

And for $P = \infty$

$$F_\infty = \left\{ \phi \in C^\infty(-\infty,\infty) : \frac{d^k\phi}{dx^k} \to 0 \ as \ x \to 0+ \ and \ x \to \infty \qquad for\ k = 0,1,2,..... \right\}$$
(5.1.6)

For $1 \le P \le \infty$, F_P is equipped with the topology generated by seminorms

$$\gamma_k{}^P(\phi) = \left\| x^k \frac{d^k\phi}{dx^k} \right\|_P$$
(5.1.7)

For any complex number μ,

$$F_{P,\mu} = \{\phi : x^{-\mu}\phi(x) \in F_P\}$$
(5.1.8)

with the topology generated by seminorms

$$\gamma_k^{P,\mu}(\phi) = \gamma_k^{P}(x^{-\mu}\phi)$$

i,e $$Y_k^{P,\mu}(\phi) = \left\| x^k \frac{d^k}{dx^k}(x^{-\mu}\phi) \right\|_P$$

(5.1.9)

The space $F'_{p,\mu}$ denoted the set of all continuous linear functionals on $F_{P,\mu}$, equipped with the topology of point wise (weak) convergence.

McBride (1977) has developed a theory of Erdelyi-kober operators on the spaces $F_{p,\mu}$ and $F'_{p,\mu}$. In (1978), he has studied the mapping properties of the Hankel transform of order v,

$$(H_v\phi)\ (x) = \int_0^\infty (xt)^{1/2} J_v(xt)\ \phi\ (t)dt$$

(5.1.10)

on the spaces $F_{P,\mu}$ and $F'_{P,\mu}$ and obtained the relations between the Erdelyi-Kober operators and modified operator of the Hankel transform on $F'_{P,\mu}$. Further in (1979), he has applied his result to a solutions of a pair of dual integral equations of Titchmarsh type.

There are mainly two approaches to extend the study of fractional calculus to generalize function. The first is based on concept of convolution of distributions. We know that the operator given by (1.4) can be expressed as convolution.

$$(I^\alpha\ f)(x) = f(x) * \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}$$

(5.1.11)

where

$$x_+^\alpha = \begin{cases} x^\alpha & if\ x > 0 \\ o & f\ x < 0 \end{cases}$$

This approach is considered by Schwartz (1950-51) and defined the fractional integer as convolution of the function $\frac{1}{\Gamma\alpha} x_{\pm}^{\alpha-1}$, with the generalized function *f*.

That is $\frac{1}{\Gamma\alpha} x_{\pm}^{\alpha-1} * f$.

Following Schwartz (1950-51), we define the fractional integral of a generalized function $f \in K'$, which is dual of test function space $K = C_0^\infty(R)$, as

$$(I^\alpha f, \phi) = \left(\frac{x_+^{\alpha-1}}{\Gamma\alpha} * f, \phi \right), \qquad \forall\ \phi \in K$$

(5.1.12)

in the case where *f* is supported on the half axis x > 0.

The second, which is more common is based on using the adjoint operator.

We know that the adjoint of the operator I^α is an operator K^α and under certain conditions we have the formula for fractional integration by parts Love and Young (1938)

$$\int_0^\infty (I^\alpha f)\ (x)\ \phi\ (x)\ dx = \int_0^\infty f(x)\ (K^\alpha\phi)\ \ (x)\ dx$$

, (5.1.13)

That is

$$< I^\alpha f, \phi > \ = \ < f, K^\alpha\phi >$$

(5.1.14)

The function *f* involving in (3.13), may indeed be defined as the generalized function if K^α maps continuously the space of test functions X into itself. When *f* and $I^\alpha f$ are considered to be generalized function on different spaces of test functions X and Y such that $f \in X'$ (the dual of the test function space X), $I^\alpha f \in Y'$ (the dual of the test function space Y), then I^α must map continuously Y onto X.

Fractional Integrals and Fractional Derivatives on the Space D′

Now following the above approach, we can define the fractional integrals and fractional derivations on the generalized function space D′.

Fractional Integral

For $f \in D'$ and $\phi \in D$, we have

$$< I^\alpha f, \phi > \ = \ \int_0^\infty (I^\alpha f)(x)\ \phi(x) dx$$

$$= \int_0^\infty \phi(x) \left\{ \frac{1}{\Gamma\alpha} \int_0^x \frac{f(t)}{(x-t)^{-\alpha+1}} \, dt \right\} dx,$$ using (1.4)

$$= \int_0^\infty f(t) \left\{ \frac{1}{\Gamma\alpha} \int_t^\infty \frac{\phi(x)}{(x-t)^{-\alpha+1}} \, dt \right\} dx$$

$$= \int_0^\infty f(t)(K^\alpha \phi)\ (t)\ dt$$, using (1.5)

$$=< f, K^\alpha \phi > \qquad (5.2.1)$$

Similarly $$< K^\alpha f, \phi > \; = \; < f, I^\alpha \phi > \qquad (5.2.2)$$

Fractional Derivation

For $f \in D'$ and $\phi \in D$, we have

$$< I^{-\alpha} f, \phi > \; = \int_0^\infty (I^{-\alpha} f)(x)\ \phi(x)\ dx,$$ using (1.6)

$$= \int_0^\infty \phi(x) \left\{ \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)}{(x-t)^{\alpha+1}} \, dt \right\} dx$$

$$= \int_0^\infty f(t) \left\{ \frac{1}{\Gamma(-\alpha)} \int_t^\infty \frac{\phi(x)}{(x-t)^{\alpha+1}} \, dx \right\} dt,$$ using (1.7)

$$= \int_0^\infty f(t)(K^{-\alpha} \phi)\ (t)\ dt$$

$$=< f, K^{-\alpha} \phi > \qquad (5.2.3)$$

Similarly we can define

$$< K^{-\alpha} f, \phi > \; = \; < f, I^{-\alpha} \phi > \qquad (5.2.4)$$

ACKNOWLEDGEMENT

I would like to acknowledge the help received from Dr. T.N. Trivedi HOD. Deptt. of Mathematics, V.S.S.D College, Kanpur.

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