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CONVERGENCE POINT FOR NONEXPANSIVE MAPPING USING NEW ITERATION IN BANACH SPACE

D.P. SHUKLA[a1], ARCHANA SINGH PARIHAR[b], SANJAYJAIN[c] AND P.K. DIXIT[d]

[a]Department of Mathematics, Govt. P.G. Science College, Rewa, M.P., India
E-mail: shukladpmp@gmail.com
[b]Department of Mathematics, Govt. P.G. Science College, Rewa, M.P., India.
E-mail: archana.rewa@gmail.com
[c]Department of Mathematics, Govt. P.G. Science College, Rewa, M.P., India
E-mail: jains@gmail.com
[d]Department of Mathematics, Govt. P.G. Girls College, Chhatarpur, M.P., India
E-mail: dixitpk@rediffmail.com

ABSTRACT

In this paper, we established new iterative methods for finding the fixed point of nonexpansive mapping in Banach space.

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We have a variety of techniques to suggest and analyze several numerical techniques for solving variational inequalities and related optimization problems. Variational inequalities, which were introduced and studied in early sixties, have played a critical and significant part in the study of several unrelated problems arising in finance, economics, network analysis, elasticity, optimization, water resources, medical images and structural analysis. Variational inequalities have witnessed an dynamic growth in theoretical advances, algorithmic development and new applications across all disciplines of pure and applied sciences and proved to productive, analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis see Xu (2002), Xu (2004) Moudafi (2000), Noor (2000) and the references there in for more details. Related to the variational inequalities, is the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis.

In Noor (2000) suggested and analyze three-step iterative method perform for finding the approximate solution of the nonexpansive mapping using the technique of updating the solution. Noor (2007) and Noor and Huang (2007) have considered some three-step iterative methods for the nonexpansive mappings in conjunction with variational inequalities.

Let X be a real Banach space with dual X* and C a nonempty closed convex subset of X. Let $J : X\, 2^{x^*}$ denote the normalized duality mapping defined by $J(x)=\{x^* \in X^* : (x,x^*) = \|x\|^2, \|x^*\| = \|x\|, x\in X\}$.

Recall that a mapping $f{:}C\rightarrow C$ is called contractive if there exists a constant $\alpha\in(0,1)$ such that $\|f(x) - f(y)\|\leq\alpha\,\|x-y\|$ x, $y\in C$. We use Π_c to denote the collection of all contractive mappings on C. Let now $T{:}C\rightarrow C$ be a nonexpansive mapping; namely, $|Tx - Ty\|\leq\|x- y\|$, for all x, $y\in C$. A point $x\in C$ is a fixed point of T provided Tx = x. Denote by F(T) the set of fixed points of T, that is, $F(T) = \{x\in C : Tx = x\}$. Throughout the paper we assume that $F(T)\neq\phi$. Halpern (1967) considered the following iterative scheme: For a given $x_0\in C$ and $u\in C$, find the approximate solution x_{n+1} by the $x_{n+1} = \alpha_n u + (1-\alpha_n)\, Tx_n$ (1)

He pointed out that both of the conditions (C1): $\lim_{n\to\infty} \alpha_n = 0$ and (C2): $\sum_{n=1}^{\infty}\alpha_n = \infty$ are necessary in the sense that if the iteration scheme (1) converges to a fixed point of T, then these conditions must be satisfied. After that, many authors considered several conditions of the iterative method (1) concerning the choice of the parameters $\{\alpha_n\}$. In particular, Xu (2004) considered the following iteration scheme

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\, Tx_n. \quad (2)$$

which is generalization of (1) and is known as the viscosity approximation method, the origin of which goes to Moudafi

[1]Corresponding author

(2000), Xu (2004) proved the strong convergence of the sequence $\{x_n\}$ by using the conditions (C1), (C2) and the following condition:

(C3) either : $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ or $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Considered the following modified Noor iteration for nonexpansive mappings.

$$w_n = \delta_n x_n + (1-\delta_n)Tx_n,$$
$$z_n = y_n x_n + (1-y_n)Tw_n,$$
$$y_n = \beta_n x_n + (1-\beta_n)Tz_n, \quad (3)$$
$$x_{n+1} = a_n u + (1-\alpha_n)y_n,$$

It is clear from (3) that modified Noor iteration include the two-step and one-step iterations as special cases of Noor iterations.

Su and Qin (2006) obtained the following result.

THEOREM SQ. Let C be a closed convex subset of a uniformly smooth Banach space X and let $T : C\to C$ be a nonexpansive mapping such that $F(T)\neq\phi$. Given a point $u\in C$, the initial guess $x_0\in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\chi_n\}$and $\{\delta_n\}$in [0,1], the following conditions are satisfied

(i) $\sum_{n=0}^{\infty}\alpha_n=\infty$, or $\lim_{n\to\infty}\alpha_n = 0$;

(ii) $\beta_n + (1+\beta_n)(1-\gamma_n)(2-\delta_n) \in (0,a)$ for some $a \in (0,1)$;

(iii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty$,

$\sum_{n=0}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty$ and $\sum_{n=0}^{\infty}|\delta_{n+1}-\delta_n|<\infty$

Then $\{X_n\}$ defined by (3) converges strongly to a fixed point of T

Motivated and inspired by the ongoing research in this direction, we consider and construct two multi-step iterations algorithms for approximating fixed pints of nonexpansive mappings. The main purpose of this paper is twofold. First we extend Su and Qin's result (2006) to a general situation with less restrictions on parameters for finding the fixed point of the nonexpansive mapping, which solve a certain variational inequality. Secondly, we propose a modified new iteration which enriches and complements the iterative methods of nonexpansive mapping.

Preliminaries and Definitions

Let X be a real Banach space with its dual X^*. Let S $=\{x \in X: \|x\|= 1\}$ denote the unit sphere of X.

The norm on X is said to be G$\hat{a}$teaux differentiable if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \quad (4)$$

exists for each x, y S and in this case X is said to be smooth. X is said to have a uniformly Frechet differentiable norm if the limit (4) is attained uniformly for x, y S and in this case X is said to be uniformly smooth. It is well-known that if X is uniformly smooth then the duality map is norm-to-norm uniformly continuous on bounded subsets of X

The first lemma is very well-known (subdifferential) inequality.

Lemma 2.1. Let X be a real Banach space and J the normalized duality map on X. Then for any given $x, y \in X$, the following inequality holds:

$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle$, $\forall j(x+y) \in J(x+y)$.

Lemma 2.2. Suzuki (2005) Let $\{x_n\}$and $\{z_n\}$be bounded sequences in a Banach space X and let $\{\alpha_n\}$be a sequence in [0, 1] which satisfies the following condition

$0 < \lim_{n\to\infty}\alpha_n \le \lim_{n\to\infty}\sup\alpha_n < 1$.

Suppose $x_{n+1} = \alpha_n x_n + (1-\alpha_n)z_n$, $n \ge 0$,

and $\lim_{n\to\infty}\sup(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|) \le 0$.

Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 2.3. Xu (2004) Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that

$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n$, $n \ge 0$,

where $\{\chi_n\}$is a sequence in (0,1) and $\{\delta_n\}$is a sequence in R such- that

(i) $\sum_{n=0}^{\infty}\gamma_n=\infty$;

(ii) $\lim_{n\to\infty}\sup\delta_n/\gamma_n \le 0$ or $\sum_{n=0}^{\infty}|\delta_n|<\infty$

Then $\lim_{n\to\infty}\to\infty\ a_n = 0$

Lemma 2.4. Xu (2004) Let C be a nonempty closed convex subset of a real uniformly Banach space X. Let $T : C\to C$ be a nonexpansive mapping with $F(T)\neq\phi$, and $f\in\prod_C$. Then $\{x_t\}$ defined by

$$x_t = tf(x_t) + (1-t)Tx_t,\ t \in [0,1] \quad (5)$$

converges strongly to a point p in F(T) which is the unique solution of the variational inequality

$<(I-f)p, j(x-p)> 0, x \in F(T).$ (6)

Lemma 2.5. Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \phi$, and $f \in \prod_C$. Given bounded sequence $\{x_n\} \subset C$ satisfying $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Then $\lim_{n\to\infty} \sup <f(z) - z, j(x_n - z)> \leq 0, z \in F(T)$.

Proof Let x_t be the unique fixed point of the contraction mapping z_t given by $z_t x = tf(x) + (1-t)Tx$

Then $x_t - x_n = t(f(x_t) - x_n) + (1-t)(Tx_t - x_n)$

We apply Lemma 2.1 to get

$\|x_t - x_n\|^2 \leq (1-t)^2 \|Tx_t - x_n\|^2 + 2t<f(x_t) - x_n, j(x_t - x_n)>$

$\leq (1-t)^2 (\|Tx_t - Tx_n\| + \|Tx_n - x_n\|)^2$

$+ 2t<f(x_t) - xt, j(x_t - x_n)> + 2t\|x_t - x_n\|^2$ (7)

$\leq (1-t)^2 \|x_t - x_n\|^2 + a_n(t) + 2t\|x_t - x_n\|^2$

$+ 2t<f(x_t) - x_t, j(x_t - x_n)>,$

where

$a_n(t) = \|Tx_n - x_n\|(2\|x_t - x_n\| + \|Tx_n - x_n\|)$ (8)

$\to 0$ as $n \to \infty$.

The last inequality (7) implies

$<x_t - f x_t, j(x_t - x_n)> \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t} a_n(t).$ (9)

It follows that $\lim_{n\to\infty} \sup <x_t - f(x_t), j(x_t - x_n)> \leq \frac{t}{2} M_1^2$,

where $M_1 > 0$ is a constant such that

$M_1 \geq \|x_t - x_n\|$ for all $t \in (0, 1)$ and $n \geq 0$

Letting $t \to 0$ in (9) yields.

$\lim_{t\to 0} \sup \lim_{n\to\infty} \sup <x_t - f(x_t), j(x_t - x_n)> \leq 0.$

Moreover, we have that

$(z - f(z), j(z - x_n))$

$= <z-f(z), j(z-x_n)> - <z-f(z), j(x_t-x_n)>$

$+ <z - f(z), j(x_t - x_n)> - <x_t - f(z), j(x_t - x_n)>$

$+ <x_t - f(z), j(x_t - x_n)> - <x_t - f(x_t), j(x_t - x_n)>$

$+ <x_t - f(x_t), j(x_t - x_n)>$

$= <z-f(z), j(z-x_n) - j(x_t-x_n)>$

$+ <z - x_t, j(x_t - x_n)> + <f(x_t) - f(z), j(x_t - x_n)>$

$+ <x_t - f(x_t), j(x_t - x_n)>.$

Then, we obtain

$\lim_{n\to\infty} \sup <z - f(z), j(z - x_n)>$

$\leq \sup_{n\in N} <z - f(z), j(z - x_n) - j(x_t - x_n)>$

$+ \|z - x_t\| \lim_{n\to\infty} \sup \|x_t - x_n\| + \|f(x_t) - f(z)\| \lim_{n\to\infty} \sup \|x_t - x_n\|$

$+ \lim_{n\to\infty} \sup <x_t - f(x_t), j(x_t - x_n)>$

$\leq \sup_{n\in N} <z - f(z), j(z - x_n) - j(x_t - x_n)>$

$+ (1 + \alpha) \|z - x_t\| \lim_{n\to\infty} \sup \|x_t - x_n\|$

$+ \lim_{n\to\infty} \sup <x_t - f(x_t), j(x_t - x_n)>.$

From Lemma 2.4, we know that $x_t \to z \in F(T)$ as $t \to 0$ and j is norm m-to-weak* uniformly continuous on bounded subset of C, we obtain

$\lim_{t\to 0} \sup_{n\in N} <z - f(z), j(z - x_n) - j(x_t - x_n)> = 0.$

Therefore we have

$\lim_{n\to\infty} \sup <z - f(z), j(z - x_n)> = \lim_{t\to 0} \sup \lim_{n\to\infty} \sup <z - f(z), j(z - x_n)>$

$\leq \lim_{t\to 0} \sup \lim_{n\to\infty} \sup <x_t - f(x_t), j(x_t - x_n)>$

≤ 0

This completes the proof.

RESULTS

Modified New Iteration

We consider and analyze another iteration for finding the approximate point of the nonexpansive mapping.

For given $x_0 \in C$, find the approximate solution x_n by the iterative scheme:

$x_{n+1} = \propto_n x_n + (1 - \propto_n) y_n,$

$y_n = \beta_n f(x_n) + (1 - \beta_n) Tz_n,$

$z_n = \gamma_n f(x_n) + (1 - \gamma_n) Tw_n$ (10)

$w_n = \delta_n f(x_n) + (1 - \delta_n) x_n$

which is called the modified new iteration. Now we state and study the convergence result of iteration scheme (10).

Theorem 3.1. Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \phi$ and $f \in \prod_C$. Given sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\chi_n\}$ and $\{\delta_n\}$ in (0,1), Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$;

(ii) $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(iii) $\lim_{n\to\infty} \delta_n = 0$ and $\sum_{n=0}^{\infty} \delta_n = \infty$;

(iv) $0 < \lim_{n\to\infty} \inf \alpha_n \le \lim_{n\to\infty} \sup \alpha_n \le 1$;

(v) $\lim_{n\to\infty} \gamma_n = 0$ and $\gamma_n/\beta_n \le M$ for some $M > 0$

(vi) $\lim_{n\to\infty} \delta_n = 0$ and $\delta_n/\beta_n \le M$ for some $M > 0$

Then, for arbitrary $x_0 \in C$, the sequence $\{x_n\}$ defined by (10) converges strongly to a fixed point $p \in F(T)$ which is the unique solution of the variational inequality (6).

Proof First we prove that $\{x_n\}$is bounded. Take p F(T), from (10), we have

$\|w_n - p\| \le \delta_n\|f(x_n) - p\| + (1 - \delta_n)\|x_n - p\|$

$\le \delta_n\|f(x_n) - f(p)\| + \delta_n\|f(p) - p\| + (1 - \delta_n)\|x_n - p\|$

$\le \alpha\delta_n \|x_n - p\| + \delta_n\|f(p) - p\| + (1 - \delta_n)\|x_n - p\|$

$\le [1-(1-\alpha)\, \delta_n]\, \|x_n - p\| + \delta_n\|f(p) - p\|$,

$\|z_n - p\| \le \gamma_n\|f(x_n) - p\| + (1 - \gamma_n)\|Tw_n - p\|$

$\le \gamma_n\|f(x_n) - f(p)\| + \gamma_n\|f(p) - p\| + (1-\gamma_n)\|w_n - p\|$

$\le \alpha\gamma_n \|x_n - p\| + \gamma_n\|f(p) - p\| + (1 - \gamma_n)\, \delta_n\|f(p) - p\|$

$+ (1 - \gamma_n)[1 - (1 - \alpha)\, \delta_n]\, \|x_n - p\|$

$\le \{\alpha\gamma_n + (1 - \gamma_n)[1 - (1 - \alpha)\delta_n]\}\|x_n - p\|$

$+ \{\gamma_n + (1 - (\gamma_n)\delta_n\}\, \|f(p) - p\|$

$\le \{1-(1-\alpha)\, [\gamma_n + (1 - \gamma_n)\delta_n]\}\|x_n - p\| + \{\gamma_n+(1-\gamma_n)\, \delta_n\}\, \|f(p)-p\|$

and hence

$\|y_n - p\| \le \beta_n\|f(x_n) - p\| + (1 - \beta_n)\|Tz_n - p\|$

$\le \beta_n\|f(x_n) - f(p)\| + \beta_n\|f(p) - p\| + (1-\beta_n)\|z_n - p\|$

$\le \alpha\beta_n \|x_n - p\| + \beta_n\|f(p) - p\| + (1 - \beta_n)\, \|z_n - p\|$

$\le \alpha\beta_n \|x_n - p\| + \beta_n\|f(p) - p\| + (1 - \beta_n)\, \{1-(1-\alpha)[\gamma_n+(1-\gamma_n)\delta_n]\}$

$\|x_n-p\| + (1-\beta_n)\{\gamma_n + (1-\gamma_n)\delta_n\}\, \|f(p) - p\|$

$\le [\alpha\beta_n + (1-\beta_n)\{1-(1-\alpha)[\gamma_n + (1 - \gamma_n)\delta_n]\}]\|x_n - p\|$

$+ [\beta_n+(1-\beta_n)\, \{\gamma_n+(1-\gamma_n)\, \delta_n\}]\, \|f(p)-p\|$

Therefore

$\|x_{n+1} - p\| \le \alpha_n\|x_n - p\| + (1 - \alpha_n)\, \|y_n - p\|$

$\le \alpha_n\|x_n - p\| + (1- \alpha_n)[\alpha\beta_n + (1-\beta_n)\{1-(1-\alpha)[\gamma_n+(1-\gamma_n)\delta_n]\}]\, \|x_n - p\|$

$+ [\beta_n + (1 - \beta_n)\{\gamma_n + (1 - \gamma_n)\delta_n\}]\, \|f(p) - p\|$

$\le \{\alpha_n + (1 - \alpha_n)\, [\alpha\beta_n + (1-\beta_n)\{1-(1-\alpha)\, [\gamma_n+(1-\gamma_n)\delta_n]\}]\}\, \|x_n - p\|$

$+ (1 - \alpha_n)[\beta_n + (1 - \beta_n)\, \{\gamma_n + (1-\gamma_n)\, \delta_n\}]\, \|f(p) - p\|$

$\le \max\{\|x_n - p\|, \|f(p) - p\|/(1-\alpha)\}$

By induction,

$\|x_n - p\| \le \max\{\|x_0 - p\|, \|f(P) - p\|/(1 - \alpha)\},\ n \ge 0$,

that is, $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{Tx_n\}$ and $\{z_n\}$

We observe that

$\|y_{n+1} - y_n\| = \|(\beta_{n+1} - \beta_n)f(x_{n+1}) + \beta_n\, (f(x_{n+1}) - f(x_n))$

$+ (1 - \beta_n)(Tz_{n+1} - Tz_n) + (\beta_n - \beta_{n+1})\, Tz_{n+1}\|$

$\le (\beta_{n+1} - \beta_n)(\|f(x_{n+1})\| + \|Tz_{n+1}\|) + \beta_n\|f(x_{n+1}) - f(x_n)\|$

$+ (1 - \beta_n)\|z_{n+1} - z_n\|$

$\le |\beta_{n+1}-\beta_n|\, (\|f(x_{n+1})+\|Tz_{n+1}\|) + \alpha\beta_n\, \|x_{n+1}-x_n\| + (1 - \beta_n)\|z_{n+1} - z_n\|,$ (11)

and

$\|z_{n+1} - z_n\| = \|(\gamma_{n+1} - \gamma_n)f(x_{n+1}) + \gamma_n(f(x_{n+1}) - f(x_n))$

$+ (1 - \gamma_n)(Tw_{n+1} - Tw_n) + (\gamma_n - \gamma_{n+1})Tw_{n+1}\|$

$\le |\gamma_{n+1} - \gamma_n|\, (\|f(x_{n+1})+\|Tw_{n+1}\|+\alpha\gamma_n\|x_{n+1}-x_n\| + (1-\gamma_n)\|w_{n+1} - w_n\|$ (12)

and also

$\|w_{n+1} - w_n\| = \|(\delta_{n+1} - \delta_n)\, f(x_{n+1}) + \delta_n(f(x_{n+1}) - f(x_n))$
$+ (1-\delta_n)(Tx_{n+1}-Tx_n) + (\delta_n - \delta_{n+1})\, Tx_{n+1}\|$

$\le |\delta_{n+1} - \delta_n|\, (\|f(x_{n+1}) + \|Tx_{n+1}\|) + \alpha\delta_n\|x_{n+1} - x_n\|$

$+ (1-\delta_n)\, \|x_{n+1}-x_n\|$ (13)

$\|w_{n+1} - w_n\| \le |\delta_{n+1} - \delta_n|\, (\|f(x_{n+1})\| + \|Tx_{n+1}\|) + \|x_{n+1} - x_n\|$

Substituting (13) into (12) that

$\|z_{n+1} - z_n\| \le |\gamma_{n+1} - \gamma_n|\, (\|f(x_{n+1})\, \|Tw_{n+1}\|) + |\delta_{n+1} - \delta_n|(\|f(x_{n+1})\|$

$+ \|Tx_{n+1}\| + \|x_{n+1} - x_n\|$ (14)

Again, substituting (14) into (11) that

$\|y_{n+1} - y_n\| \le |\beta_{n+1} - \beta_n|\, (\|f(x_{n+1})\| + \|Tz_{n+1}\|) + |\gamma_{n+1} - \gamma_n|(\|f(x_{n+1})+ \|Tw_{n+1}\|)$

$+ |\delta_{n+1} - \delta_n|\, (\|f(x_{n+1})\| + \|Tz_{n+1}\|) + \|x_{n+1} - x_n\|$ (15)

It follows that from $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \delta_n = 0$ and (15) that

$\lim_{n\to\infty} \sup(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$

Hence, by Lemma 2.2, we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \quad (16)$$

Then

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\alpha_n)\|y_n - x_n\| = 0. \quad (17)$$

From (10), we obtain

$$\|x_n - Tx_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - Tx_n\|$$

$$\le \|x_{n+1} - x_n\| + \alpha_n\|x_n - Tx_n\| + (1-\alpha_n)\|y_n - Tx_n\|$$

$$\le \|x_{n+1} - x_n\| + \alpha_n\|x_n - Tx_n\| + (1-\alpha_n)\beta_n\|f(x_n) - Tx_n\|$$

$$+ (1-\alpha_n)(1-\beta_n)\|Tz_n - Tx_n\| \quad (18)$$

$$\|x_n - Tx_n\| \le \|x_{n+1} - x_n\| + \alpha_n\|x_n - Tx_n\| + (1-\alpha_n)\beta_n\|f(x_n) - Tx_n\|$$

$$+ (1-\alpha_n)(1-\beta_n)\gamma_n\|f(x_n) - x_n\|$$

We note that $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$ and $\{x_n\}$, $\{f(x_n)\}$ and $\{Tx_n\}$ are all bounded, therefore from (18) with (17), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \quad (19)$$

Since

$$\|y_n - Ty_n\| \le \|y_n - Tx_n\| + \|Tx_n - Ty_n\|$$

$$\le \|y_n - x_n\| + \|x_n - Tx_n\| + \|y_n - x_n\|$$

$$\|y_n - Ty_n\| = 2\|y_n - x_n\| + \|x_n - Tx_n\|. \quad (20)$$

It follows from (16), (19) and (20) that

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0. \quad (21)$$

Again by similar way

$$\|z_n - Tz_n\| \le \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Tz_n\|$$

$$\le y_n\|f(x_n) - x_n\| + \|x_n - y_n\| + \beta_n\|f(x_n) - Tz_n\|$$

which implies that (noting that (i) and (v) and (16)

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0 \quad (22)$$

$$\|w_n - Tw_n\| \le \|w_n - x_n\| + \|x_n - y_n\| + \|y_n - Tw_n\|$$

$$\le \delta_n\|f(x_n) - x_n\| + \|x_n - y_n\| + \beta_n\|f(x_n) - Tw_n\|$$

from (i), (iii), (vi) and (16), we have

$$\lim_{n\to\infty} \|w_n - Tw_n\| = 0 \quad (23)$$

It follows from (21), (22) and (23) and lemma (2.5) that

$$\left.\begin{array}{l} \lim\sup_{n\to\infty} \langle f(p) - p, j(y_n - p)\rangle \le 0 \text{ and} \\ \lim\sup_{n\to\infty} \langle f(p) - p, j(z_n - p)\rangle \le 0 \\ \lim\sup_{n\to\infty} \langle f(p) - p, j(w_n - p)\rangle \le 0 \end{array}\right\} \quad (24)$$

From (10) and lemma (2.1), we have

$$\|w_n - p\|^2 = \|\delta_n(f(x_n) - p) + (1-\delta_n)(x_n - p)\|^2$$

$$\le (1-\delta_n)^2\|x_n - p\|^2 + 2\delta_n\langle f(x_n) - p, j(w_n - p)\rangle$$

$$\le (1-\delta_n)^2\|x_n - p\|^2 + 2\delta_n\langle f(x_n) - f(p), j(w_n - p)\rangle$$

$$+ 2\delta_n\langle f(p) - p, j(w_n - p)\rangle$$

$$\le (1-\delta_n)^2\|x_n - p\|^2 + 2\alpha\delta_n(\|x_n - p\|, \|w_n - p\|$$

$$+ 2\delta_n\langle f(p) - p, j(w_n - p)\rangle$$

$$\le (1-\delta_n)^2\|x_n - p\|^2 + \alpha\delta_n(\|x_n - p\|^2 + \|w_n - p\|^2)$$

$$+ 2\delta_n\langle f(p) - p, j(w_n - p)\rangle$$

That is

$$\|w_n - p\|^2 \le \left[1 - \frac{2(1-\alpha)\delta_n}{1-\alpha\delta_n}\right]\|x_n - p\|^2 \; \frac{\delta_n^2}{1-\alpha\delta_n}\|x_n - p\|^2$$

$$+ \frac{2\delta_n}{1-\alpha\delta_n}\langle f(p) - p, j(w_n - p)\rangle \quad (25)$$

and

$$\|z_n - p\|^2 = \|\gamma_n(f(x_n) - p) + (1-\gamma_n)(Tw_n - p)\|^2$$

$$\le (1-\gamma_n)^2\|Tw_n - p\|^2 + 2\gamma_n\langle f(x_n) - p, j(z_n - p)\rangle$$

$$\le (1-\gamma_n)^2\|w_n - p\|^2 + 2\gamma_n\langle f(x_n) - f(p), j(z_n - p)\rangle$$

$$+ 2\gamma_n\langle f(p) - p, j(z_n - p)\rangle$$

$$\le (1-\gamma_n)^2\|w_n - p\|^2 + 2\alpha\gamma_n\|x_n - p\|, \|z_n - p\|$$

$$+ 2\gamma_n\langle f(p) - p, j(z_n - p)\rangle$$

$$\le (1-\gamma_n)^2\|w_n - p\|^2 + \alpha\gamma_n(\|x_n - p\|^2 + \|z_n - p\|^2)$$

$$+ 2\gamma_n \le f(p) - p, j(z_n - p)\rangle$$

$$\le (1-\gamma_n)^2\left[\left[1 - \frac{2(1-\alpha)\delta_n}{1-\alpha\delta_n}\right]\|x_n - p\|^2 \; \frac{\delta_n^2}{1-\alpha\delta_n}\|x_n - p\|^2 + \frac{2\delta_n}{1-\alpha\delta_n}\langle f(p) - p, j(w_n - p)\rangle\right]$$

$$+ \alpha\gamma_n\|x_n - p\|^2 + \alpha\gamma_n\|z_n - p\|^2 + 2\gamma_n\langle f(p) - p, j(z_n - p)\rangle$$

$$\|z_n - p\|^2 \le \left[1 - \frac{2(1-\alpha)\gamma_n}{1-\alpha\gamma_n}\right]\|x_n - p\|^2 +$$

$$\left[\frac{\gamma_n^2}{1-\alpha\gamma_n} + \frac{\delta_n^2}{(1-\alpha\gamma_n)(1-\alpha\delta_n)}\right]$$

$$\|x_n - p\|^2 + \frac{2\delta_n}{(1-\alpha\gamma_n)(1-\alpha\delta_n)}\langle f(p) - p, j(w_n - p)\rangle$$

$$+ \frac{2\gamma_n}{(1-\alpha\gamma_n)} < f(p) - p, j(z_n - p)> \quad (26)$$

and

$||y_n - p||^2 = ||\beta_n(f(x_n) - p + (1 - \beta_n)(Tz_n - p||^2$

$\leq (1-\beta_n)^2||Tz_n - p||^2 + 2\beta_n<f(x_n) - p, j(y_n-p)>$

$\leq (1-\beta_n)^2||Tz_n - p||^2 + 2\beta_n<f(x_n) - f(p), j(y_n-p)>$

$+ 2\beta_n <f(p) - p, j(y_n-p)>$

$\leq (1-\beta_n)^2||z_n - p||^2 + 2\alpha\beta_n (||x_n - p||, ||y_n - p||)$

$+ 2\beta_n <f(p) - p, j(y_n-p)>$

$\leq (1-\beta_n)^2||z_n - p||^2 + \alpha\beta_n(||x_n-p||^2 + ||y_n - p||^2)$

$+ 2\beta_n <f(p) - p, j(z_n - p)>$

$$\leq (1-\beta_n)^2\left[\left[1-\frac{2(1-\alpha)\gamma_n}{1-\alpha\gamma_n}\right]||x_n - p||^2 \left[\frac{\gamma_n^2}{(1-\alpha\gamma_n)}+\frac{\delta_n^2}{(1-\alpha\gamma_n)(1-\alpha\delta_n)}\right]||x_n - p||^2 + \frac{2\delta_n}{(1-\alpha\gamma_n)(1-\alpha\delta_n)} < f(p) - p, j(w_n - p)> + \frac{2\gamma_n}{1-\alpha\gamma_n} <f(p) - p, j(z_n - p> \right] +$$

$\alpha\beta_n (||x_n-p||^2 + ||y_n - p||^2)$

$+2\beta_n < f(p) - p, j(y_n - p)>$

$$||y_n - p||^2 \leq \left[1-\frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n}\right]||x_n - p||^2 + \left[\frac{\beta_n^2}{(1-\alpha\beta_n)}+\frac{\gamma_n^2}{(1-\alpha\beta_n)(1-\alpha\gamma_n)} + \frac{\delta_n^2}{(1-\alpha\beta_n)(1-\alpha\gamma_n)(1-\alpha\delta_n)}\right]||x_n - p||^2 +$$

$$\frac{2\delta_n}{(1-\alpha\gamma_n)(1-\alpha\delta_n)(1-\alpha\beta_n)} < f(p) - p, j(w_n - p)> + \frac{2\gamma_n}{(1-\alpha\gamma_n)(1-\alpha\beta_n)} < f(p) - p, j(z_n - p)> + \frac{2\beta_n}{(1-\alpha\beta_n)} < f(p) - p, j(y_n - p)> \quad (27)$$

Again from (10)

$||x_{n+1} - p||^2 = ||\alpha_n (x_n - p + (1 - \alpha_n) (y_n - p)||^2$

$\leq [\alpha_n||x_n-p||+(1-\alpha_n) ||y_n - p||]^2$

$\leq \alpha_n^2 ||x_n - p||^2 +(1-\alpha_n)^2||\gamma_n-p||^2 + 2\alpha_n(1-\alpha_n)||x_n - p|| \, ||y_n - p||$

$\leq \alpha_n^2 ||x_n - p||^2 +(1-\alpha_n)^2||\gamma_n-p||^2 + \alpha_n(1-\alpha_n)(||x_n - p||^2 + ||y_n - p||^2)$

$= \alpha_n||x_n-p||^2 +(1-\alpha_n) ||y_n - p||^2 \quad (28)$

Substituting (26), (27) into (28), we have

$$||x_{n+1} - p||^2 \leq \left[1-\frac{2(1-\alpha)(1-\alpha_n)\beta_n}{1-\alpha\beta_n}\right]||x_n - p||^2 + \left[\frac{(1-\alpha_n)\beta_n^2}{1-\alpha\beta_n}+\frac{(1-\alpha_n)\gamma_n^2}{(1-\alpha\beta_n)(1-\alpha\gamma_n)} + \frac{(1-\alpha_n)\delta_n^2}{(1-\alpha\beta_n)(1-\alpha\gamma_n)(1-\alpha\delta_n)}\right]$$

$$||x_n - p||^2 + \frac{2(1-\alpha_n)\delta_n}{(1-\alpha\delta_n)(1-\alpha\gamma_n)(1-\alpha\beta_n)} < f(p) - p, j(w_n - p)>$$

$$+ \frac{2(1-\alpha_n)\gamma_n}{(1-\alpha\gamma_n)(1-\alpha\beta_n)} < f(p) - p, j(z_n - p)> + \frac{2(1-\alpha_n)\beta_n}{(1-\alpha\beta_n)} < f(p) - p, j(y_n - p)>$$

$= (1 - \delta_n) ||x_n - p||^2 + \delta_n\sigma_n \quad (29)$

where $\delta_n = \dfrac{2(1-\alpha)(1-\alpha_n)\beta_n}{(1-\alpha\beta_n)}$

$$\sigma_n = \left\{\left[\frac{\beta_n}{2(1-\alpha)}+\frac{\gamma_n^2}{2(1-\alpha)(1-\alpha\gamma_n)\beta_n} + \frac{\delta_n^2}{2(1-\alpha)(1-\alpha\delta_n)(1-\alpha\gamma_n)\beta_n}\right] M_2 + \frac{\delta_n}{(1-\alpha\delta_n)(1-\alpha\gamma_n)(1-\alpha)\beta_n} < f(p) - p, j(w_n - p)> + \frac{\gamma_n}{(1-\alpha)(1-\alpha\gamma_n)\beta_n} < f(p) - p, j(z_n - p)> + \frac{1}{(1-\alpha)} < f(p) - p, j(y_n - p)> \right\}$$

where $M_2 > 0$ is a constant such that $\|x_n - p\|^2 \leq M_2$ for all $n \geq 0$. It is easily seen from (i)-(vi) and (24) that

$$\sum_{n=0}^{\infty} \delta_n = \infty \text{ and } \limsup_{n\to\infty} \sigma_n \leq 0$$

Finally apply Lemma (2.3) to (29) and conclude that $x_n \to p$. This completes the proof.

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