MODIFIED VARIATIONAL ITERATION METHOD FOR SOLVING DUFFING EQUATIONS

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ABSTRACT

In this paper, an efficient modification of He's variational iteration method (VIM) is presented. Since, achieving accurate approximations by the first few iterations of VIM, for Duffing equations is not possible, therefore, we suggest to approximate the integrand involved, by Lagrange interpolation polynomial using Chebyshev points. Numerical examples are worked out and the results show that the approximate solutions get better by increasing the number of points.

KEYWORDS: Variational iteration method, Duffing equation, Lagrange interpolation polynomials, Chebyshev points

Variational iteration method is proposed by He (He, 1997, 1998, 1999, 2000, 2004, 2006) as a modification of a general Lagrange multiplier method (Inokuti et al. 1978). Over the years, this method has been applied for solving delay differential equations (He, 1997), Helmholtz equation (Abuasad and Momani, 2006), Burgers and coupled Burgers equations (Abdou and Soliman, 2005), differential algebraic equations (Soltanian et al. 2009), and many other problem (Abdou and Soliman, 2005; Abdou, 2007; Dehghan and Tatari, 2007).

The purpose of this paper is to introduce a modification of variational iteration method using optimal Lagrange interpolation polynomials for solving Duffing equations.

The Duffing equation is a nonlinear equation of applied science. A general form of this equation is as follows:

\[ Lu(x) + Nu(x) = g(x) \]  (2)

where, \( L \) is a linear operator, \( N \) is a nonlinear operator, and \( g \) is a known analytic function.

According to VIM, we can construct the following correction functional

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x, t) \{ Lu_n(t) + N u_n(t) - g(t) \} dt \]  (3)

where, \( \lambda \) is a Lagrange multiplier (He, 2000), which can be identified optimally via the variation theory, and \( \delta u_n = 0 \) is a restricted variation which means \( \delta u_n = 0 \).

The successive approximations, \( u_n(x) ; \ n \geq 1 \), of the solution \( u(x) \) will be readily obtained upon using the obtained Lagrange multiplier and by using selective function \( u_0(x) \).

Consequently, the exact solution maybe obtained by using

\[ u(x) = \lim_{n \to \infty} u_n(x) \]

when \( u_n(x) \) has a limit as \( n \to \infty \).

In fact, the solution of problem (2) is considered as a
fixed point of the following functional under a suitable choice of the initial term \( u_0(x) \):

\[
u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x, t)(Lu_n(t) + Nu_n(t) - g(t))dt.
\]

(4)

As a powerful tool, we recall Banach's theorem:

**Theorem 1: (Banach's fixed point theorem)**

Assume that \( X \) is a Banach space and

\[ A: X \rightarrow X \]

Is a nonlinear mapping, and suppose that

\[ ||A[u] - A[v]|| \leq \gamma ||u - v||, \quad u, v \in X, \]

for some constant \( \gamma < 1 \).

Then \( A \) has a unique fixed point. Furthermore, the sequence

\[ u_{n+1} = A[u_n], \]

with an arbitrary choice of \( u_0 \in X \), converges to the fixed point of \( A \).

According to Theorem 1, for the nonlinear mapping

\[ A[u(x)] = u(x) + \int_0^x \lambda(x, t)(Lu(t) + Nu(t) - g(t))dt \]

A sufficient condition for convergence of the variational iteration method, is strict contraction of \( A \).

Furthermore, the sequence (4) converges to the fixed point of \( A \) which is also the solution of problem (2).

**The modified variational iteration method for solving (1) (MVIM)**

In general, we can apply only a few iterations of VIM for Duffing equations, because as we precede the integrand involved on the right hand side of (3) becomes complicated.

Therefore, to obtain a high accuracy solution we replace the integrand by optimal Lagrange interpolation polynomials as follows

\[
f(x) = p(x) = \sum_{k=0}^{n} f(x_k) \prod_{j=0, k \neq j}^{n} \frac{x-x_j}{x_k-x_j}\]

(5)

where \( x_k \)'s are the roots of the \((n+1)\)st chebyshev polynomial of the first kind \( T_{n+1}(x) \) in \([-1,1]\).

**Remark**

For ensuring uniform convergence, a better choice of interpolation points is the set of zeros of the chebyshev polynomials \( T_{n+1}(x) \) (6).

**Proof:** see (Handscomb and Mason, 2003).

Therefore, we put \( x_k \) in (5) as follows:

\[ x_k = \frac{a + b}{2} + \frac{b - a}{2} \cos \left( \frac{2k+1}{2n+2} \pi \right) k = 0, 1, ..., n \]

(6)

**Numerical Examples**

In this section, we use modification of the VIM which is introduced in section 3 for three nonlinear Duffing equations. All of the calculation have been done with Maple 15 with 6 digits precision. In all of examples we put \( n=20, 25, 30 \) in (6), and 10 iterations of VIM have been used.

**Example 1**

Consider the following nonlinear Duffing equation

\[
\begin{cases}
\ddot{u}(x) + u(x) + u^3(x) = \cos^3(x) - \sin(x) \\
u(0) = 1, \quad u'(0) = 1
\end{cases}
\]

(7)

with the exact solution \( u(x) = \cos(x) \).

According to VIM, we can construct the correction functional of equation (7) as follows:

\[
u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left( u_n(t) + \ddot{u}_n(t) + \ddot{u}_n(t) + \ddot{u}_n(t) - \cos^3(t) + \sin(t) \right)dt
\]

(8)

where \( \lambda \) is general Lagrange multiplier and \( \ddot{u}_n(t), \ddot{u}_n(t), \ddot{u}_n(t), \ddot{u}_n(t) \) denote restricted variation, i.e.

\[
\delta \ddot{u}_n(t) = \delta \ddot{u}_n(t) = \delta \ddot{u}_n(t) = \delta \ddot{u}_n(t) = 0
\]

The stationary conditions yields:

\[ 1 - \lambda \left( x, t \right) \big|_{t=x} = 0, \lambda \left( x, t \right) \big|_{t=x} = 0 \]

Therefore, the Lagrange multiplier can be identified as:

\[ \lambda = t - x \]

hence, the following iteration formula is obtained:

\[
u_{n+1}(x) = u_n(x) + \int_0^x (t-x)(u_n(t) + u_n(t) + u_n(t) + u_n(t) - \cos^3(t) + \sin(t))dt
\]

According to equation (7), initial approximation is
and numerical results are tabulated in Table 1, where by
\( u_m(x) \) we mean mth iteration of (9).

Using ordinary VIM, \( u_4(x) \), takes nearly 10 minutes to
be calculated partly and the computer gives the warning
"Computation Interrupted."

This means that we cannot obtain more than two
element for this example!

Table 1. Comparison of \(|u(x) - u_{11}(x)|\) for example 1
using MVIM with 20, 25, 30 Chebyshev points.

<table>
<thead>
<tr>
<th></th>
<th>MVIM</th>
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Example 2

Consider the following Duffing equation

\[
\begin{cases}
u''(x) + 2u'(x) + u(x) + 8u^3(x) = e^{-3x} \\
u(0) = \frac{1}{2}, \quad u'(0) = -\frac{1}{2}
\end{cases}
\]

with the exact solution \( u(x) = \frac{1}{2}e^{-x} \).

Similar to example 1 the iteration formula for equation (10) is:

\[ u_{n+1}(x) = u_n(x) + \int_0^x (t-x)\left( u_n(t) + 2u_n(t) + u_n(t) + 8u_n(t) e^{-3t} \right) dt \] (11)

According to (10) initial approximation is

\[ u_0(x) = \frac{1}{2} - \frac{1}{2}x \]

And the numerical results are tabulated in Table 2.

For example if we use ordinary VIM for (11), in the second iteration, \( u_2(x) \) is as follows:

\[
0.001066xe^{-6x} - 0.000411xe^{-6x} + 0.001371x^4 e^{-6x} - 0.002743x^3 e^{-6x} - 0.001143x^2 e^{-6x} - 0.000003xe^{-6x} - 0.014391x e^{-6x} - 0.02781e^{-6x} - 0.263031x^2 e^{-3x} - 0.13580x^6 e^{-3x} - 0.001646x^7 e^{-3x} - 0.279428x e^{-3x} - 0.0295267x^3 - 0.168952x^4 e^{-3x} - 0.000740x^{10} e^{-3x} + 0.000246x^9 e^{-3x} + 0.470198 - 0.216820x^4 - 0.072067x^5 - 0.032718e^{-3x} - 0.040740x^6 - 0.120438x^7 - 0.0000246x^10 - 0.024511x^{11} + 0.020025x^{12} + 0.009123x^{13} + 0.002692x^{14} - 0.0050523x^{15}.
\]

It is clear that, next iterations become more complicated, hence to overcome this difficulty we approximate the integrand by Lagrange interpolation polynomial.

Table 2. Comparison of \(|u(x) - u_{11}(x)|\) for example 2
using MVIM with 20, 25, 30 Chebyshev points.

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Example 3

Consider the following Duffing equation

\[
\begin{align*}
\ddot{u}(t) + 3u(t) - 2u^3(t) &= \cos(u(t)) \sin(2u(t)) \\
u(0) &= 0, \quad u'(0) = 1
\end{align*}
\]

with the exact solution \(u(x) = \sin(x)\).

The iteration formula for example 3 according to VIM is as follows:

\[
u_{n+1}(x) = \nu_n(x) + \int_0^x (t-x)(\nu_n(t) + 3\nu_n(t) - 2\nu_n^3(t) - \cos(\nu_n(t)) \sin(2\nu_n(t))) dt\]

According to equation (12) initial approximation is \(u_0(x) = x\) and the numerical results are tabulated in Table 3.

Table 3. Comparison of \(|u(x) - u_{11}(x)|\) for example 3 using MVIM with 20, 25, 30 Chebyshev points.

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CONCLUSION

In this paper, we presented a modification of VIM to solve Duffing equations. This modification is based on replacing the integrand, involved in the corresponding correction functional, by using optimal Lagrange interpolation. Numerical experience show that using proposed modification one can obtain more accurate approximate solutions.

REFERENCES


