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CERTAIN EXPANSION OF BASIC HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

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ABSTRACT

In this paper, certain expansions involving generalized basic Hypergeometric Series of two variables have been established. These expansions include, as special cases, many of known expansions of basic hypergeometric functions.

KEYWORDS: Basic hypergeometric functions, well poised q-analogue expansions

A basic hypergeometric function(Agarwal,1959) of two variables is defined as:

$$\phi\left[\begin{matrix}(a):(c);(d); \\ (b):(e);(f);\end{matrix} q,x,y\right] = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{((a;q))_{s+t}((c;q))_s((d;q))_t x^s y^t}{((b;q))_{s+t}(e;q))_s((f;q))_t(q;q)_a\,(q;q)_t}$$

$$(|x|<1, |y|<1, |q|<1)$$

Where $(\alpha;q)(\alpha;q)_n = (1-\alpha)(1-\alpha q)(1-\alpha q^2)\ldots(1-\alpha q^{n-1})$

$$(\alpha;q)_0 = 1;\quad (a;q)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{a^n\,(q/a;q)_n}$$

MAIN RESULTS

In this section, following results have been established(Agarwal.1953):

$$\sum_{r=0}^{\infty} \frac{((a;q))_{2r}(\alpha;q)_r\ 2\alpha-\beta;q)_r(\alpha;q)_r\,(\alpha-\frac{1}{2};q)_r x^r y^r q^{u(r)}}{((d;q))_{2r}(q;q)_r(\alpha+r-1;q)_r(2\alpha;q)_{2r}}$$

$$\times\emptyset\left[\begin{matrix}(a)+2r:\alpha+r\ ,2\alpha-\beta+r,\alpha+r,\alpha+\frac{1}{2}+r; \\ (d)+2r:\ 2\alpha+2r\qquad ;2\alpha+2r\qquad ;\end{matrix} xq^{\beta-\alpha-1/2,y}\right]$$

$$=\emptyset\left[\begin{matrix}(a),\alpha:2\alpha-\beta;\alpha+\frac{1}{2}; \\ (d),2\alpha:\ -\ ;\ -\ ;\end{matrix}\ xq^{\beta-\alpha-1/2},y\right] \qquad \ldots(1)$$

where , $u(r) = r(\beta+r-1)$

$$\sum_{r=0}^{\infty} \frac{((a;q))_r(\alpha-1;q)_r(\delta;q)_r(1-\beta;q)_r x^r y^r q^{r(1-\alpha-\delta)}}{((d;q))_r(q;q)_r(\alpha+\delta-\beta;q)_r(\alpha-\beta;q)_r}$$

$$\times\left[\begin{matrix}(a)+r\quad ;1-\beta+r:1-\beta;b; \\ (d)+r,\quad \alpha-\beta+r:\qquad ;e\end{matrix}\ xq^{\beta-\alpha-\delta},y\right]$$

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$$=\emptyset\left[\begin{matrix}(a),1-\beta+\delta,1-\beta; & b; & \\ (d),\alpha-\beta+\delta: & ;e\ ; & \end{matrix} xq^{\beta-\alpha-\delta},y\right] \qquad ...(2.)$$

$$\sum_{r=0}^{\infty}\frac{((a;q))_{2r}(b-e;q)_r\,(b-c;q)_r(e;q)_r\,(e+c-b;q)_r x^r y^r q^{u(r)}}{(q;q)_r(b+r-1;q)_r((d;q))_{2r}(b;q)_{2r}}$$

$$\times\emptyset\left[\begin{matrix}(a)+2r: b-e+r\ ,b-c+r,b-e+r,e+c-b+r; & \\ (d)+2r:\ b+2r & ;b+2r\quad ;\end{matrix} xq^{e+c-b},y\right]$$

$$=\emptyset\left[\begin{matrix}(a),b-e:b-;e+c-b; \\ (d),b:\ :\ -\ ;\quad ;-;\end{matrix}\ xq^{e+c-b},y\right] \qquad ...(3)$$

Where , $u(r)=r(c+r-1)$

PROOF OF RESULTS

To prove (2) , left hand side can be written as:

$$\sum_{r=0}^{\infty}\frac{((a;q))_{2r}(\alpha;q)_r\,(2\alpha-\beta;q)_r(\alpha;q)_r\,(\alpha-\frac{1}{2};q)_r x^r y^r q^{u(r)}}{((d;q))_{2r}(q;q)_r(2\alpha+r-1;q)_r(2\alpha;q)_{2r}}$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a+2r;q))_{m+n}(\alpha+r;q)_m(2\alpha-\beta+r;q)_m(\alpha+r;q)_n}{((d+2r;q))_{m+n}(2\alpha+2r;q)_m(2\alpha+2r;q)_n}\times\frac{(\alpha+\frac{1}{2}+r;q)_n x^m q^{m(\beta-\alpha-\frac{1}{2})}y^n}{(q;q)_m(q;q)_n}$$

$$=\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a;q))_{2r+m+n}(2\alpha;q)_{2r}(\alpha-\frac{1}{2};q)_r}{((d;q))_{2r+m+n}(q;q)_r(2\alpha+r-1;q)_r(\alpha+\frac{1}{2};q)_r}$$

$$\times\frac{(\alpha;q)_{r+m}(2\alpha-\beta;q)_{r+m}(\alpha;q)_{r+n}(\alpha+\frac{1}{2};q)_{r+n}x^{r+m}y^{r+n}q^{m(\beta-\alpha-\frac{1}{2})}}{(2\alpha;q)_{2r+m}(2\alpha;q)_{2r+n}(q;q)_m(q;q)_n}$$

Which on putting $r+m=s$ and $r+n=t$,becomes

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{((a;q))_{s+t}(\alpha;q)_s(2\alpha-\beta;q)_s(\alpha;q)_t(\alpha+\frac{1}{2};q)_r x^s y^t q^{s(\beta-\alpha-\frac{1}{2})}}{((d;q))_{s+t}(2\alpha;q)_s(2\alpha;q)_t(q;q)_s(q;q)_t}$$

$$\times 4\emptyset_3\left[\begin{matrix}2\alpha-1,-\frac{2\alpha+1}{2},q^{-s};q^{-t}; & \\ -\frac{2\alpha-1}{2},2\alpha q^{s}, \quad 2\alpha q^{t} & \end{matrix} q^{\alpha+s+t+1/2}\right]$$

Now summing inner $4\emptyset_3$ by Dixon theorem (Slater,1966) the above expansion becomes:

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{((a;q))_{s+t}(\alpha;q)_s(2\alpha-\beta;q)_s(\alpha;q)_t(\alpha+\frac{1}{2};q)_r x^s y^t q^{s(\beta-\alpha-\frac{1}{2})}}{((d;q))_{s+t}(2\alpha;q)_s(2\alpha;q)_t(q;q)_s(q;q)_t}\times\frac{(\alpha q^s;q)_t}{(2\alpha q^s;q)_t}$$

Which is equal to right hand side of (2).

To prove (2), we write the left hand side as

$$\sum_{r=0}^{\infty}\frac{((a;q))_r(\alpha-1;q)_r\,(\delta;q)_r(1-\beta;q)_r\,x^r\,q^{(1-\alpha-\beta)r}.y^r}{((d;q))_r(q;q)_r(\alpha+\delta-\beta;q)_r(\alpha-\beta;q)_r}$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a+r;q))_{m+n}(1-\beta+r;q)_m(1-\beta;q)_m(b;q)_n}{((d+r;q))_{m+n}(\alpha-\beta+r;q)_m(e;q)_n}\ \frac{x^m q^{m(\beta-\alpha-\delta)}y^n}{(q;q)_m(q;q)_n}$$

$$=\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a;q))_{r+m+n}(\alpha-1;q)_r(\delta;q)_r(\alpha-\frac{1}{2};q)_r}{((d;q))_{r+m+n}(q;q)_r(\alpha+\delta-\beta;q)_{r+m}(e;q)_n}$$

$$\times\frac{(1-\beta;q)_m(b;q)_n x^{r+m}y^{r+n}q^{m(\beta-\alpha-\delta)}}{(q;q)_m(q;q)_n}$$

Which after taking $r+m=s$ and $r+n=t$,equals

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{((a;q))_{s+t}(1-\beta;q)_s(b;q)_t(1-\beta;q)_s\ q^{s(\beta-\delta-\alpha)}x^s y^t}{((d;q))_{s+t}\,(\alpha-\beta;q)_s\,(e;q)_t(q;q)_s\,(q;q)_t}\times 3\emptyset_2\left[\begin{matrix}\alpha-1, & \delta, & q^{-s}; & \\ \alpha+\delta-\beta, & \beta q^{-s}; & \beta q^{-s}\ ; & \end{matrix} q\right]$$

After some simplification..

Now summing inner $3\emptyset_2$ with the help of q-analogue of Saalschutz theorem (Slater,1966)and(Verma,1964), the above expansion becomes :

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{((a;q))_{s+t}(1-\beta;q)_s(b;q)_t(1-\beta;q)_s\, q^{s(\beta-\delta-\alpha)}x^s y^t}{((d;q))_{s+t}\,(\alpha-\beta;q)_s\,(e;q)_t(q;q)_s\,(q;q)_t}\times\frac{(\beta-\alpha+1-s;q)_s(\beta-\delta-s;q)_s}{(\beta-s\,;q)_s(\beta-\alpha-\delta+1+s;q)_s}$$

Which after some simplification, yields the right hand side of (2).

To prove (2) , we write its left hand side as:

$$\sum_{r=0}^{\infty}\frac{((a;q))_{2r}\,(b-e;q)_r\,(b-c;q)_r(e+c-b;q)_r\,x^r y^r q^{u(r)}}{(d;q)_r(q;q)_r(b+r-1;q))_r(b;q)_{2r}}$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a+2r;q))_{m+n}(b-e+r;q)_m(b-c+r;q)_m(b-e;q)_n(e+c-b;q)_n}{((d+2r;q))_{m+n}(b+2r;q)_m(b+2r;q)_n}\times\frac{x^m y^n q^{m(e+c-b)}}{(q;q)_m(q;q)_n}$$
$$=\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a;q))_{2r+m+n}(b-e;q)_{r+m}(b-c;q)_{r+m}(e;q)_r}{((d;q))_{2r+m+n}(q;q)_r(b+r-1;q)_r(b-e;q)_r}$$
$$\times\frac{(b-e;q)_{r+n}(e+c-b;q)_{r+n}x^{r+m}y^{r+n}}{((b;q))_{2r+m}(b;q)_{2r+n}(q;q)_m(q;q)_n}\,q^{r(r+1)+r(c-r)+m(e+c-b)}$$

Which on taking $r+m=s$ and $r+n=t$ becomes

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{((a;q))_{s+t}(b-c;q)_s(b-e;q)_s(b-e;q)_t(e+c-b;q)_t}{((d;q))_{s+t}(b;q)_s(b;q)_t(q;q)_s}\times\frac{x^s y^t q^{s(e+c-b)}}{(q;q)_t}$$
$$\times\sum_{r=0}^{\infty}\frac{(b;q)_{2r}(e;q)_r(q^{-s};q)_r(q^{-t};q)_r}{(q;q)_r(b+r-1;q)_r(b-e;q)_r(bq^t;q)_r}\,q^{r(b-e+s+t)}.$$
$$=\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{((a;q))_{s+t}(b-c;q)_s(b-e;q)_s(b-e;q)_t(e+c-b;q)_t}{((d;q))_{s+t}(b;q)_s(b;q)_t(q;q)_s}\times\frac{x^s y^t q^{s(e+c-b)}}{(q;q)_t}$$
$$\times\, 6\emptyset_5\begin{bmatrix}b-1\,,\frac{b+1}{2}-\frac{b+1}{2},e\,,q^{-s},q^{-t};\;q^{b-e+s+t}\\ \frac{b-1}{2},-\frac{b-1}{2},b-e,bq^s,\quad bq^t\,;\end{bmatrix}$$

Now, summing well poised $6\emptyset_5$ with the help of known result (Chaundy,1942)),we get right side of (2).

SPECIAL CASES:

(i) If we let $q\to 1$ in the results(1) to (3) , we get corresponding expansions for ordinary Hypergeometric function of two variables.(Burchnall, et al., 1940)

$$\sum_{r=0}^{\infty}\frac{((a))_{2r}(\alpha)_r(2\alpha-\beta)_r(\alpha)_r(\alpha-\frac{1}{2})_r x^r y^r}{((d))_{2r}(1)_r(\alpha+r-1)_r(2\alpha)_{2r}(\alpha+\frac{1}{2})_r}\times F\begin{bmatrix}(a)+2r\colon\alpha+r,2\alpha-\beta+r;\alpha+r,\alpha+\frac{1}{2}+r;\;x,y\\(d)+2r\colon 2\alpha+2r;2\alpha+2r;\end{bmatrix}$$
$$=F\begin{bmatrix}(a),\alpha\colon 2\alpha-\beta;\alpha+\frac{1}{2}+r;\;x,y\\(d),2\alpha;-\quad;-;\end{bmatrix}\qquad\ldots(4)$$

$$\sum_{r=0}^{\infty}\frac{((a))_r(\alpha-1)_r(\delta)_r(1-\beta)_r x^r y^r}{((d))_r(1)_r(\alpha+\delta-\beta)_r(\alpha-\beta)_r}\times F\begin{bmatrix}(a)+r;1-\beta+r,1-\beta;b;\;x,y\\(d)+r\colon\alpha-\beta+r;e;\end{bmatrix}$$
$$=F\begin{bmatrix}(a)\colon 1-\beta+\delta,1-\beta;b;\;x,y\\(d)\colon\alpha-\beta+\delta;e;\end{bmatrix}\qquad\ldots(5)$$

$$\sum_{r=0}^{\infty}\frac{((a))_{2r}(b-e)_r(b-c)_r(e)_r(e+c-b)_r x^r y^r}{((1))_r(b+r-1)_r((d))_{2r}(b)_{2r}}\times$$
$$F\begin{bmatrix}(a)+2r\colon b-e+r,b-c+r;b-e+r,e+c-b+r;\;x,y\\(d)+2r\colon b+2r;b+2r\end{bmatrix}$$

$$=F\begin{bmatrix}(a), b-e: b-c; e+c-b; & x, y\\(d), b: -; -;\end{bmatrix} \quad ...(6)$$

Case (ii) Putting $A = D = 0, y = x\ in\ (6)$, we get-

$$\sum_{r=0}^{\infty}\frac{(e;q)_r(b-e;q)_r(b-c;q)_r(e+c-b;q)_r x^{2r}q^{r(c+r-1)}}{(q;q)_r(b+r-1;q)_r(b;q)_{2r}}\times 2\emptyset_1\begin{bmatrix}b-e+r, b-c+r; & xq^{e+c-b}\\b+2r;\end{bmatrix}$$

$$\times 2\emptyset_1\begin{bmatrix}b-e+r, e+c-b; & x\\b+2r;\end{bmatrix}$$

$$=\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(b-e;q)_{s+t}(b-c;q)_s(e+c-b;q)_t x^{s+t}q^{s(e+c-b)}}{(b;q)_{s+t}(q;q)_s(q;q)_t} \quad ... (7)$$

Now, letting $s + t = v$ in right hand side of (7),it is equal to

$$=\sum_{s=0}^{\infty}\sum_{v=0}^{\infty}\frac{(b-e;q)_v\,(b-c;q)_s(e+c-b;q)_{v-s}x^v q^{s(e+c-b)}}{(b;q)_v(q;q)_s(q;q)_{v-s}}$$

$$=\sum_{v=0}^{\infty}\frac{(b-e;q)_v(e+c-b;q)_v x^v}{(b;q)_v(q;q)_v}\sum_{s=0}^{\infty}\frac{(b-c;q)_s(e+c-b+v;q)_{-s}\,xq^{s(e+c-b)}}{(q;q)_s(1+v;q)_{-s}}$$

$$=\sum_{v=0}^{\infty}\frac{(b-e;q)_v(e+c-b;q)_v x^v}{(b;q)_v(q;q)_v}\times 2\emptyset_1 F\begin{bmatrix}b-c, q^{-v}; & q\\1+b-e-c-v;\end{bmatrix}$$

Again summing the inner $2\emptyset_1$ by the basic analogue of Gauss theoem (Slater,1966) we have

$$\sum_{r=0}^{\infty}\frac{(e;q)_r(b-e;q)_r(b-c;q)_r(e+c-b;q)_r x^{2r}q^{r(c+r-1)}}{(q;q)_r(b+r-1;q)_r(b;q)_{2r}}$$

$$2\emptyset_1\begin{bmatrix}b-e+r, b-c+r; & xq^{e+c-b}\\b+2r;\end{bmatrix}\times 2\emptyset_1\begin{bmatrix}b-e+r, e+c-b+r; & x\\b+2r;\end{bmatrix}= 2\emptyset_1\begin{bmatrix}e, b-e; & x\\b;\end{bmatrix} \quad ...(8)$$

Which is equivalent to a result due to (Jackson, 1942 and1944)

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