COLLOCATION METHOD FOR VOLterra-Fredholm Integro-DIFFerential EQUATIONS

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ABSTRACT

This paper introduces an approach for obtaining the numerical solution of the linear and nonlinear Volterra-Fredholm integro-differential equations based on quintic B-spline functions. The solution is collocated by quintic B-spline and then the integrand is approximated by 5-points Gauss-Turan quadrature formula with respect to the Legendre weight function. The main characteristic of this approach is that it reduces linear and nonlinear Volterra-Fredholm integro-differential equations to a system of algebraic equations, which greatly simplifying the problem. The error analysis of proposed numerical method is studied theoretically. Numerical examples illustrate the validity and applicability of the proposed method.

KEYWORDS: Volterra-Fredholm, integro-differential equations, quintic B-spline, Gauss-Turan quadrature formula, Error analysis

AMS Subject Classifications 41A15, 65R20

Consider the nonlinear Volterra-Fredholm integro-differential equation of the form

\begin{equation}
\sum_{r=0}^{m} p_r(t) y^{(r)}(t) = g(t) + \int_{a}^{t} k_1(t,x,y(x))dx + \int_{a}^{b} k_2(t,x,y(x))dx, \quad 1 \leq m \leq 4, \quad t \in [a,b],
\end{equation}

with the boundary conditions,

\begin{equation}
\sum_{r=0}^{m-1} \left[q_{i,r} y^{(r)}(a) + \beta_{i,r} y^{(r)}(b) \right] = \gamma_i, \quad 0 \leq i \leq m - 1,
\end{equation}

where \( q_{i,r}, \beta_{i,r} \) and \( \gamma_i \) are given real constants. The given kernels \( k_1, k_2 \) are continuous on \([a, b]\) and satisfy a uniform Lipschitz, and \( g(t) \) and \( p_r(t) \) are the known functions and \( y \) is unknown function. The boundary value problems in terms of integro-differential equations have many practical applications. A physical event can be modelled by the differential equation, an integral equation or an integro-differential equation or a system of these equations. Some of the phenomena in physics, electronics, biology, and other applied sciences lead to nonlinear Volterra-Fredholm integro-differential equations. Of course, these equations can also appear when transforming a differential equation into an integral equation [1-7].

In general, nonlinear Volterra-Fredholm integro-differential equations do not always have solutions which we can obtain using analytical methods. In fact, many of real physical phenomena encountered are almost impossible to solve by this technique. Due to this, some authors have proposed numerical methods to approximate the solutions of nonlinear Fredholm-Volterra integro-differential equations. To mention a few, in [8] the authors have discussed the Taylor polynomial method for solving integro-differential equations (1). The triangular functions method has been applied to solve the same equations in [9]. Furthermore,

The operational matrix with block-pulse functions method is carried out in [10] for the aforementioned integro-differential equations. The Hybrid Legendre polynomials and Block-Pulse functions approach for solving integro-differential equations (1) are proposed in [11]. Yalcinbas in [12] developed the Taylor polynomial solutions for the nonlinear Volterra-Fredholm integral equations and in [13] considered the high-order linear Volterra-Fredholm integro-differential equations. The numerical solvability of Fredholm and Volterra integro-differential equations and other related equations can be found in [14-28]. Using a global approximation to the solution of Fredholm and Volterra integral equation of

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the second kind is constructed by means of the spline quadrature in [32-37].

In this paper we will develop a collocation method based on quintic B-spline to approximate the unknown function in equation (1) then the 5-points Gauss-Turán quadrature formula with respect to the Legendre weight function is used to approximate the nonlinear Volterra-Fredholm integro-differential equations of second kind.

The organization of the paper is as follows. In Section 2, we describe the basic formulation of the quintic B-spline collocation method then the Gauss-Turán quadrature formula is discussed in Section 3. Section 4 devoted to the solution of Eq. (1) by using collocation method for Volterra-Fredholm integro-differential equation. In section 5, description of new approaches for the error analysis of the schemes is given. Finally numerical examples are given in section 6 to illustrate the efficiency of the presented method.

Quintic B-spline

We introduce the quintic B-spline space and basis functions to construct an interpolation $s$ to be used in the formulation of the quintic B-spline collocation method. Let $\pi: \{a = t_0 < t_1 < \cdots < t_N = b\}$, be a uniform partition of the interval $[a, b]$ with step size $h = \frac{b-a}{N}$. The quintic spline space is denoted by

$$S_5(\pi) = \{s \in C^4[a, b]; s|[t_i, t_{i+1}] \in P_5, \quad i = 0, 1, \ldots, N\},$$

where $P_5$ is the class of quintic polynomials. The construction of the quintic B-spline interpolate $s$ to the analytical solution $y$ for (1) can be performed with the help of the ten additional knots such that $t_{-5} < t_{-4} < t_{-3} < t_{-2} < t_{-1}$ and $t_{N+1} < t_{N+2} < t_{N+3} < t_{N+4} < t_{N+5}$.

Following [33] we can define a quintic B-spline $s(t)$ of the form

$$s(t) = \sum_{i=-2}^{N+2} c_i B_i^5(t),\quad (3)$$

where

$$B_i(t) = \frac{1}{120h^5} \begin{cases} (t - t_{i-3})^5, & t \in [t_{i-3}, t_{i-2}] \\ (t - t_{i-3})^5 - 6(t - t_{i-2})^5, & t \in [t_{i-2}, t_{i-1}] \\ (t - t_{i-3})^5 - 6(t - t_{i-2})^5 + 15(t - t_{i-1})^5, & t \in [t_{i-1}, t_i] \\ (t - t_{i-3})^5 - 6(t - t_{i-2})^5 + 15(t - t_{i-1})^5 - 20(t - t_i)^5, & t \in [t_i, t_{i+1}] \\ (t - t_{i-3})^5 - 6(t - t_{i-2})^5 + 15(t - t_{i-1})^5 - 20(t - t_i)^5 + 15(t - t_{i+1})^5, & t \in [t_{i+1}, t_{i+2}] \\ (t - t_{i-3})^5 - 6(t - t_{i-2})^5 + 15(t - t_{i-1})^5 - 20(t - t_i)^5 + 15(t - t_{i+1})^5 - 6(t - t_{i+2})^5, & t \in [t_{i+2}, t_{i+3}] \\ 0, & \text{otherwise} \end{cases}$$

Satisfying the following interpolator conditions:

$$s(t_i) = y(t_i), \quad 0 \leq i \leq N,$$

and the end conditions

$$(i) s^1(t_0) = y^1(t_0), \quad s^1(t_N) = y^1(t_N), \quad j = 1, 2,$$

or

$$(ii) D^j s(t_0) = D^j s(t_N), \quad j = 1, 2, 3, 4,\quad (4)$$

or

$$(iii) s^1(t_0) = 0, s^1(t_N) = 0, \quad j = 3, 4.$$

On quadrature formulae of Gauss–Turán

Let $P_m$ be the set of all algebraic polynomials of degree at most $m$. The Gauss-Turán quadrature formula in [29,31] is
\[
\int_a^b f(x) d\lambda(x) = \sum_{i=0}^{2n} \sum_{v=1}^n A_{i,v} f^{(i)}(\tau_v) + R_{n,2s}(f), \tag{5}
\]

where \( n \in \mathbb{N}, s \in \mathbb{N}_0 \) and \( d\lambda(x) \) is a nonnegative measure on the interval \((a, b)\) which can be the real axis \( \mathbb{R} \), with compact or infinite support for which all moments:

\[
\mu_k = \int_a^b x^k d\lambda(x), \quad k = 0, 1, \ldots
\]

exist and are finite, moreover \( \mu_0 > 0 \), and \( A_{i,v} = \int_a^b \psi_{i,v}(x) d\lambda(x) \) \((i = 0, \ldots, 2s, v = 1, \ldots, n)\) and \( \psi_{i,v}(x) \) are the fundamental polynomials of Hermite interpolation. The nodes \( \tau_v (v = 1, \ldots, n) \) in (5) are the zeros of polynomial \( \pi_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \) which minimizes the integral

\[
F(a_0, a_1, \ldots, a_{n-1}) = \int_a^b [\pi_n(x)]^{2s+2} d\lambda(x), \tag{6}
\]

then the formula (5) is exact for all polynomials of degree at most \( 2(s+1)n - 1 \), that is, \( R_{n,2s}(f) = 0, \forall f \in P_{2(s+1)n-1} \). The condition (6) is equivalent with the following conditions:

\[
\int_a^b [\pi_n(x)]^{2s+1} x^k d\lambda(x) = 0, \quad (k = 0, \ldots, n - 1) 
\]

let \( \pi_n(x) \) denoted by \( P_{n,s} \) and \( d\lambda(x) = w(x) \) on \([a, b]\). In this article, we use 5-points Gauss–Turán quadrature formula with respect to the weight function Legendre \( w(x) = 1 \) and \([-1, 1]\) with \( n = 5, s = 3 \), therefore we can approximate integral as

\[
\int_{-1}^1 f(x) d(x) = \sum_{i=0}^{6} \sum_{v=1}^5 A_{i,v} f^{(i)}(\tau_v) + R_{5,6}(f), \tag{7}
\]

\[
\int_{-1}^1 [\pi_5(x)]^7 x^k d(x) = 0, \quad (k = 0, 1, 2, 3, 4), \tag{8}
\]

where \( \pi_5(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \), by solving system (8) we can obtain \( a_i \) \((i = 0, 1, 2, 3, 4)\) coefficients, on the other hand we have

\[
\pi_{v+1}(x) = (x - \alpha_v) \pi_v(x) - \beta_v \pi_{v-1}(x), \quad v = 0, 1, 2, 3, 4, \\
\pi_{-1}(x) = 0, \quad \pi_0(x) = 1,
\]

\[
\text{Where}
\]

\[
\alpha_v = \alpha_0(3,5) = \frac{(x \pi_v, \pi_v)}{\pi_v, \pi_v} = \frac{\int_{-1}^1 x \pi_v^2(x) \pi_n^2 dx}{\int_{-1}^1 \pi_v^2(x) \pi_n^2 dx} = \frac{\int_{-1}^1 x \pi_v^2(x) \pi_5^2 dx}{\int_{-1}^1 \pi_v^2(x) \pi_5^2 dx}, \\
\beta_v = \beta_0(3,5) = \frac{\pi_v, \pi_v}{\pi_{v-1}, \pi_{v-1}} = \frac{\int_{-1}^1 \pi_v^2(x) \pi_{v-1}^2 dx}{\int_{-1}^1 \pi_{v-1}^2(x) \pi_{v-1}^2 dx} = \frac{\int_{-1}^1 \pi_v^2(x) \pi_5^2 dx}{\int_{-1}^1 \pi_{v-1}^2(x) \pi_5^2 dx}, \\
\beta_0 = \int_{-1}^1 \pi_5^2 dx,
\]

so that we can obtain the zeros of polynomial \( \pi_5^{3,5}(x) \) of eigenvalue Jacobian matrix
Finally to determine $A_{i,v}$, we use the following polynomial for approximation of function $f(x)$

$$ f_{k,v}(x) = (x - \tau_v)^k \Omega_v(x) = (x - \tau_v)^k \prod_{i=0}^n (x - \tau_i)^{2s+1}, \quad (9) $$

where $0 \leq k \leq 2s$, $1 \leq v \leq n$ and

$$ \Omega_v(x) = \left( \frac{\pi_v(x)}{x - \tau_v} \right)^{2s+1} = \prod_{i=0}^n (x - \tau_i)^{2s+1}, \quad 1 \leq m \leq 4, t \in [a,b], \quad (11) $$

The approximate solution of nonlinear Volterra-Fredholm integro-differential equation

In the given nonlinear Volterra-Fredholm integro-differential Eq. (1), we can approximate the unknown function by quintic B-spline (3), then we obtain:

$$ \sum_{r=0}^m p_r(t) s^{(r)}(t) = g(t) + \int_a^t k_1(t,x,s(x))dx + \int_a^b k_2(t,x,s(x))dx, \quad 1 \leq m \leq 4, t \in [a,b], \quad (11) $$

with the boundary conditions,
By solving the above nonlinear system, we determine the coefficients $c_i, i = -2, \ldots, N + 2$, by setting $c_i$ in (3), we obtain the approximate solution for Equation (11).

**Error Analysis**

In this section, we consider the error analysis of the Volterra-Fredholm integro-differential equation of the second kind. To obtain the error estimation of our approximation, first we recall the following definitions in [29-31, 33].

**Definition 5.1** The Gauss-Turán quadrature formula with multiple nodes,
\[
\int_a^b f(x) \, dx = \sum_{i=0}^{2n} \sum_{\nu=1}^n A_{1,\nu} f^{(i)}(\tau_{\nu}) + R_{n,2s}(f),
\]

is exact for all polynomials of degree at most \(2(s + 1)n - 1\), that is, \(R_{n,2s}(f) = 0 \forall f \in P_{(s+1)n-1}\).

**Definition 5.2** The most immediate error analysis for spline approximates \(S\) to a given function \(f\) defined on an interval \([a, b]\) follows from the second integral relations.

If \(f \in C_c[a, b]\), then \(\|D^j (f - S)\| \leq y h^{j+1}, j = 0, \ldots, 6\). Where \(\|f\|_\infty = \max_{0 \leq i \leq N} \sup_{t \leq t_i \leq t_{i+1}}|f(t)|\), and \(D^j\) the \(j\)-th derivative.

**Theorem 5.1** The approximate method

\[
\sum_{r=0}^{m} p_r(t_j) s^{(r)}(t_j) = g(t_j) + \frac{h}{2} \sum_{p=0}^{j-1} \sum_{\nu=1}^{5} \sum_{l=0}^{6} A_{1,\nu} k_1^{(l)}(t_j, \xi_{p\nu} s(\xi_{p\nu})) + \frac{h}{2} \sum_{p=0}^{N-1} \sum_{\nu=1}^{5} \sum_{l=0}^{6} A_{1,\nu} k_2^{(l)}(t_j, \xi_{p\nu} s(\xi_{p\nu})),
\]

\[
\sum_{r=0}^{m} \left[ a_{1r} s^{(r)}(a) + \beta_{1r} s^{(r)}(b) \right] = \gamma_j, \quad 0 \leq i \leq m - 1, j = 1, \ldots, N,
\]

for solution of the Volterra-Fredholm integro-differential Eq. (11) is converge and the error bounded is

\[
\left| e^{(m)}_j \right| \leq \frac{hL}{2|p_m|} \sum_{p=0}^{j-1} \sum_{\nu=1}^{5} \sum_{l=0}^{6} |A_{1,\nu}| \left| e^{(m)}_j \right| + \frac{hL^*}{2|p_m|} \sum_{p=0}^{N-1} \sum_{\nu=1}^{5} \sum_{l=0}^{6} |A_{1,\nu}| \left| e_{1p\nu} \right| + \frac{1}{|p_m|} \sum_{r=0}^{m-1} \left| p_{rj} \right| \left| e_{2p\nu} \right|.
\]

**Proof:** We know that at \( t_j = a + jh, h = \frac{b-a}{N}, j = 0, 1, \ldots, N\), the corresponding approximation method for nonlinear Volterra-Fredholm integro-differential Eq. (11) is

\[
\sum_{r=0}^{m} p_r(t_j) s^{(r)}(t_j) = g(t_j) + \frac{h}{2} \sum_{p=0}^{j-1} \sum_{\nu=1}^{5} \sum_{l=0}^{6} A_{1,\nu} k_1^{(l)}(t_j, \xi_{p\nu} s(\xi_{p\nu})) + \frac{h}{2} \sum_{p=0}^{N-1} \sum_{\nu=1}^{5} \sum_{l=0}^{6} A_{1,\nu} k_2^{(l)}(t_j, \xi_{p\nu} s(\xi_{p\nu})),
\]

\( j = 1, \ldots, N, \quad 1 \leq m \leq 4\). (17)

By discrditing (1) and approximating the integral by the 5-points Gauss–Tura’n rules, we obtain

\[
\sum_{r=0}^{m} p_r(t_j) y^{(r)}(t_j) = g(t_j) + \frac{h}{2} \sum_{p=0}^{j-1} \sum_{\nu=1}^{5} \sum_{l=0}^{6} A_{1,\nu} k_1^{(l)}(t_j, \xi_{p\nu} y(\xi_{p\nu})) + \frac{h}{2} \sum_{p=0}^{N-1} \sum_{\nu=1}^{5} \sum_{l=0}^{6} A_{1,\nu} k_2^{(l)}(t_j, \xi_{p\nu} y(\xi_{p\nu})),
\]

\( j = 1, \ldots, N, \quad 1 \leq m \leq 4\). (18)

By subtracting (18) from (17) and using interpolatory conditions of quintic B-spline, we get

\[
\sum_{r=0}^{m} p_r(t_j) [s^{(r)}(t_j) - y^{(r)}(t_j)]
\]

\[
= \frac{h}{2} \sum_{p=0}^{j-1} \sum_{\nu=1}^{5} \sum_{l=0}^{6} A_{1,\nu} k_1^{(l)}(t_j, \xi_{p\nu} s(\xi_{p\nu})) - k_1^{(l)}(t_j, \xi_{p\nu} y(\xi_{p\nu})) + \frac{h}{2} \sum_{p=0}^{N-1} \sum_{\nu=1}^{5} \sum_{l=0}^{6} A_{1,\nu} k_2^{(l)}(t_j, \xi_{p\nu} s(\xi_{p\nu})) - k_2^{(l)}(t_j, \xi_{p\nu} y(\xi_{p\nu})),
\]

So
\[ p_m(t_j) \| s^{(m)}(t_j) - y^{(m)}(t_j) \| \\
\leq \frac{h}{2} \sum_{p=0}^{j-1} \sum_{v=1}^{5} \sum_{l=0}^{6} |A_{1v}||k_1(t_j, \xi_{pu}, s(\xi_{pu}))| \\
- k_1(t_j, \xi_{pu}, y(\xi_{pu})) + \frac{h}{2} \sum_{p=0}^{N-1} \sum_{v=1}^{5} \sum_{l=0}^{6} |A_{1v}||k_2(t_j, \xi_{pu}, s(\xi_{pu}))| - k_2(t_j, \xi_{pu}, y(\xi_{pu})) \\
+ \sum_{r=0}^{m-1} |p_r(t_j) \| s^{(r)}(t_j) - y^{(r)}(t_j) \|, j = 1, \ldots, N. \]

We suppose that \( s^{(m)}(t_j) = s_j^{(m)}, y^{(m)}(t_j) = y_j^{(m)}, j = 1, \ldots, N, m = 1,2,3,4 \) and kernels \( k_1, k_2, l = 0, \ldots, 6 \) satisfy a Lipschitz condition in its third argument of the form

\[ |k_1(t, \xi, s) - k_1(t, \xi, y)| \leq L|s - y|, \]
\[ |k_2(t, \xi, s) - k_2(t, \xi, y)| \leq L^*|s - y|, \]

where \( L, L^* \) is independent of \( t, \xi, s \) and \( y \). We get

\[ |p_m| \| s_j^{(m)} - y_j^{(m)} \| \leq \frac{hL}{2} \sum_{p=0}^{j-1} \sum_{v=1}^{5} \sum_{l=0}^{6} |A_{1v}||s_{pu} - y_{pu}| + \frac{hL^*}{2} \sum_{p=0}^{N-1} \sum_{v=1}^{5} \sum_{l=0}^{6} |A_{1v}||s_{pu} - y_{pu}| + \sum_{r=0}^{m-1} \|p_{rj} \| \| s_{j}^{(r)} - y_{j}^{(r)} \|, j = 1, \ldots, N. \]

Since that \( |p_m| \| \neq 0 \), then we have

\[ |e_j^{(m)}| \leq \frac{hL}{2|p_m|} \sum_{p=0}^{j-1} \sum_{v=1}^{5} \sum_{l=0}^{6} |A_{1v}||e_{1pu}e_j^{(m)}| + \frac{hL^*}{2|p_m|} \sum_{p=0}^{N-1} \sum_{v=1}^{5} \sum_{l=0}^{6} |A_{1v}||e_{2pu}e_j^{(m)}| + \sum_{r=0}^{m-1} \|p_{rj} \| \| e_{j}^{(r)} \|. \]

Where \( e_j^{(m)} = s_j^{(m)} - y_j^{(m)}, j = 1, \ldots, N, r = 0, \ldots, m \).

When \( h \to 0 \) then the above first and second term are zero and the third term in the above tends to zero because this term is due to interpolating of \( y(t) \) by quintic B-spline. We get for a fixed \( j \),

\[ |e_j^{(m)}| \to 0 \text{ as } h \to 0, m = 0, \ldots, 4. \]

**Numerical Examples**

In order to test the applicability of the presented method, we consider two examples of the nonlinear Volterra - Fredholm integro-differential equations, these examples have been solved with various values of \( N \). The absolute errors in the solution for various values of \( N \) are tabulated in Tables. The tables verified that our approach is more accurate. Programs are preformed by Mathematica for all the examples.

**Example 6.1** Consider the following nonlinear Volterra-Fredholm integro-differential equation with the exact solution \( y(t) = t^2 \),

\[ y'(t) + y(t) = -\frac{1}{2} \int_0^t y^2(x)dx + \frac{1}{4} \int_0^t x y^2(x)dx + g(t), \quad 0 \leq x, t \leq 1, \]

where \( g(t) = \frac{1}{10} t^6 + t^2 + 2t - \frac{1}{32}, \) with boundary conditions: \( y(0) = 0 \).

This equation has been solved by our method with \( N = 6, 16, 46 \), the absolute error at the particular grid points is tabulated in table 2 . Table 3 shows, a comparison between the absolute errors of our method together with triangular functions method [9], operational matrix with block-pulse functions method [10], and Hybrid Legendre polynomials and block-pulse functions.
method [11] and reproducing kernel Hilbert space method [38]. As it is evident from the comparison results, it was found that our method in comparison with the mentioned methods is better with a view to accuracy and utilization.

**Example 6.2** Consider the following nonlinear Volterra-Fredholm integro-differential equation with the exact solution $y(t) = -1 + t^2$,

$$y'''(t) + y(t) = \int_0^t y^2(x) \, dx + \int_0^1 (t^2 x + tx^2) \, y^2(x) \, dx + g(t), \quad 0 \leq x, t \leq 1,$$

where $g(t) = \frac{-1}{5} t^5 + \frac{2}{3} t^3 + \frac{5}{6} t^2 - \frac{113}{105} t - 1$, with boundary conditions: $y(0) = -1, y'(0) = 0, y''(0) = 2$.

The approximate solutions are calculated for different values of $N = 5, 15, 35$, the absolute error at the particular grid points is tabulated in table 4. While in [39], the adopted method yields no solution for $N = 3$, takes time to give the approximate answer for $N = 5$, and is proper only for $N = 4$, our method is significant not only because it yields solutions for any $N$, but also because the approximate solutions it gives are very close to exact ones. This table verified that our results are more accurate.

**Table 2: The error ||E|| in solution of example 6.1 at particular points**

<table>
<thead>
<tr>
<th>t</th>
<th>N=6</th>
<th>N=16</th>
<th>N=46</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.67E-19</td>
<td>1.08E-19</td>
<td>1.35E-20</td>
</tr>
<tr>
<td>0.1</td>
<td>1.48E-09</td>
<td>2.01E-10</td>
<td>2.42E-11</td>
</tr>
<tr>
<td>0.2</td>
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</tr>
<tr>
<td>0.3</td>
<td>3.85E-09</td>
<td>5.45E-10</td>
<td>6.60E-11</td>
</tr>
<tr>
<td>0.4</td>
<td>4.87E-09</td>
<td>6.89E-10</td>
<td>8.35E-11</td>
</tr>
<tr>
<td>0.5</td>
<td>5.75E-09</td>
<td>8.14E-10</td>
<td>9.86E-11</td>
</tr>
<tr>
<td>0.6</td>
<td>6.43E-09</td>
<td>9.14E-10</td>
<td>1.11E-10</td>
</tr>
<tr>
<td>0.7</td>
<td>6.90E-09</td>
<td>9.84E-10</td>
<td>1.19E-10</td>
</tr>
<tr>
<td>0.8</td>
<td>7.11E-09</td>
<td>1.01E-09</td>
<td>1.22E-10</td>
</tr>
<tr>
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<td>1</td>
<td>6.20E-09</td>
<td>8.91E-10</td>
<td>1.08E-10</td>
</tr>
</tbody>
</table>
Table 3: Numerical comparison of absolute error for Example 6.1

<table>
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</thead>
<tbody>
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<td>m = 32</td>
<td>m = 32</td>
<td>n = 8, m = 8</td>
<td>N = 26, n = 5</td>
<td></td>
</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>7.60E−11</td>
<td>1.66E−04</td>
<td>2.18E−03</td>
<td>3.10E−05</td>
<td>6.14E−07</td>
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Table 4: The error ||E|| in solution of example 6.2 at particular points

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CONCLUSION

In the present work, a technique has been developed for solving the linear and nonlinear Volterra-Fredholm integro-differential equations by using the 5-points Gauss-Turan quadrature formula with respect to the Legendre weight function and collocating by quintic B-spline. These equations are converted to a system of linear or nonlinear algebraic equations in terms of the linear combination coefficients appearing in the representation of the solution in spline basic functions. This method tested on 2 examples. The absolute errors in the solutions of these examples show that our
approach is more accurate in comparison with the methods given in [9,10,11,38,39] and our results verified the accurate nature of our method.

REFERENCES


