ANALYSIS OF THIN FILM FLOW OF A FOURTH GRADE FLUID ON AN INCLINED PLANE

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ABSTRACT

In this paper, the steady uniform and unidirectional thin film flow of an incompressible fourth grade fluid on a plane inclined at an angle $\alpha \neq 0$ to the horizontal has been taken into consideration utilizing three methods, that is, Collocation Method (CM), Galerkin Method (GM), and Least Square Method (LSM). The results generated from CM, GM and LSM are compared employing a numerical method (NUM) for validation. The approximate results of LSM in compare to CM and GM, for greater regions of dimensionless non-Newtonian parameter $B$ and for all possible boundary conditions were found to be in acceptable agreement with numerical solutions. It was observed that increasing the non-Newtonian parameter $B$ for the boundary condition $U'(1) > 0$ caused velocity profiles to increase and for the boundary condition $U'(1) = 0$ caused velocity profiles to decrease.

KEYWORDS: Fourth grade fluid, Inclined plane, Collocation Method (CM), Galerkin Method (GM), Least Square Method (LSM).

In recent years, the study of non-Newtonian fluids has gained great importance, and this is mainly due to their huge range of industrial and technological applications (Siddiqui et al., 2010). The classical Navier–Stokes equations have been proved insufficient to describe and illustrate the characteristics of complex rheological fluids such as shampoo, blood, synovial, food stuffs, paints, micro fluidics and polymer solutions. These kinds of fluids are generally known as non-Newtonian fluids. Many empirical and semi-empirical non-Newtonian models or constitutive equations have been proposed (Islam et al. 2011). Among these, the fluids of differential type (Dunn and Rajagopal, 1995; Truesdell and Noll, 2004) have received considerable attention.

The flow of thin film on an inclined plane has drawn the attention of studies in the last five decades due to its various applications in technological development of modern science. These films are susceptible to different types of instabilities due to the influence of various physical factors (e.g. gravity, viscosity, mean surface tension, thermo capillarity) (Neossi Nguetchue, 2009). A number of investigations have been carried out to analyze the flow of thin film on an inclined plane using various analytical methods (Siddiqui et al., 2006; Hayat et al., 2008; Kumaran et al. 2012; Hashmi et al., 2013). Some simple and accurate approximation techniques for solving differential equations called the Weighted Residuals Methods (WRMs), such as, Least Square (LSM), Galerkin (GM) and Collocation (CM) Methods (Stern and Rasmussen, 1996; Hatami and Ganji, 2013; Sheikholeslami et al., 2013; Hatami et al., 2013; Necati Özisik, 1993).

Motivated by these facts, we used CM, GM and LSM to obtain the solutions of the steady uniform and unidirectional thin film flow of an incompressible fourth grade fluid on an inclined plane. What is more, the corresponding results are compared and verified with that found by a type of numerical analysis as Boundary Value Problem (BVP) (Ascher et al., 1995) obtained by Maple software.

FORMULATION OF THE PROBLEM

Consider the steady uniform and unidirectional thin film flow of an incompressible fourth grade fluid down a plane inclined at an angle $\alpha \neq 0$ to the horizontal. We assumed that the ambient air was stationary, surface tension negligible and thin film of uniform thickness $\delta$. We used a $(x,y)$ coordinate system, where $x$ is in the direction of motion of the fluid on the plate and the $y$-axis is perpendicular to the plate (Fig. 1).

The basic equations governing the motion of an incompressible fluid, neglecting the thermal effects and absence of pressure gradient, are (Mohyuddin, 2005)
\[ \nabla \cdot V = 0, \quad (1) \]
\[ \rho \frac{DV}{Dt} = \rho B + \text{div}\, \tau, \quad (2) \]

where \( \nabla \) is the Nabla operator, \( V \) the velocity vector, \( \rho \) the constant density, \( B \) the body force, \( \tau \) the stress tensor, and \( \text{D/Dt} \) denotes the material derivative. As discussed in (Islam et al. 2011), the stress tensor \( \tau \) defining a fourth-grade fluid is given by

\[ \tau = \sum_{i=1}^{4} S_i \tau_i, \quad (3) \]

where
\[ S_0 = -pI, \]
\[ S_1 = \mu A_1, \]
\[ S_2 = \alpha_1 A_2 + \alpha_2 A_2^2, \]
\[ S_3 = \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (r \tau (A_1)) A_1, \]
\[ S_4 = \gamma_1 A_4 + \gamma_2 (A_1 A_2 A_3 + A_1 A_4 + A_2 A_4 + A_3 A_4) + \gamma_3 (\tau \tau (A_1)) A_4 + \gamma_4 (\tau \tau (A_2)) A_2 + \gamma_5 (\tau \tau (A_3)) A_3 + \gamma_6 (\tau \tau (A_4)) A_4, \]

where \( I \) is the identity tensor, \( \mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7 \) and \( \gamma_8 \) are material constants.

The Rivlin-Ericksen kinematic tensors \( A_n \) (Siddiqui et al., 2009; Mohyuddin, 2005) are defined as, \( A_0 = I \) and

\[ A_n = \frac{DA_{n-1}}{dt} + A_{n-1} (\nabla V) + (\nabla V)^\top A_{n-1}, \quad n \geq 1, \quad (5) \]

in which \( t \) is the transpose.

Since the flow is one-dimensional, we have

\[ V = (u(y), 0, 0). \quad (6) \]

By using the written code in Maple, the stress tensor, \( \tau \) for differential type fluid flow is obtained. After substituted it into Eq. (2), for steady one-dimensional flow of a fourth grade fluid, we have:

\[ \mu \frac{d^2 u}{dy^2} + 6\beta \left( \frac{du}{dy} \right)^2 = 0, \quad (7) \]

where \( \beta = \beta_2 + \beta_3 \) and \( A = \rho g \sin \alpha \). Therefore, the problem reduces to solve the second-order nonlinear ordinary differential equation (7) subject to the following condition due to no slip condition at inclined plate and no shearing at free surface:

\[ u(y) = 0 \text{ at } y = 0 \quad (8) \]
\[ \tau_{xy} = \mu \frac{du}{dy} + 2\beta \left( \frac{du}{dy} \right)^3 = 0 \text{ at } y = \delta \quad (9) \]

Eq. (7) and Eq. (9) indicate that not only the momentum equation of fourth grade fluids for the above problem, but also its boundary conditions is different from the ones for Newtonian fluids which are due to the different shear stress tensor. For universal use, we now write the boundary value problem (7)–(9) in terms of a set of dimensionless variables as follow

\[ \eta = \frac{y}{d}, \quad U(\eta) = \frac{\mu u(y)}{Ad^2}, \quad B = \frac{A^2 d^2 \beta}{\mu^3} \quad (10) \]

so that in non-dimensional form, Eq. (7) becomes

\[ \frac{d^2 U}{d\eta^2} + 6B \left( \frac{dU}{d\eta} \right)^2 \frac{d^2 U}{d\eta^2} + 1 = 0, \quad (11) \]

and the corresponding conditions given by Eq. (8) becomes

\[ U(\eta) = 0 \text{ at } \eta = 0, \quad (12) \]

and Eq. (9) after solving for \( \frac{du}{dy} \bigg|_{y=\delta} \) becomes

\[ \frac{dU}{d\eta} \bigg|_{\eta=1} = 0 \quad (13) \]
\[ \frac{dU}{d\eta} \bigg|_{\eta=1} = \frac{1}{\sqrt{2B}}, \quad B < 0 \quad (14) \]
\[ \frac{dU}{d\eta} \bigg|_{\eta=1} = -\frac{1}{\sqrt{2B}}, \quad B < 0. \quad (15) \]

Now, we have three different boundary conditions on the free surface of the thin film fluid. However regions and signs of non-Newtonian parameters for second and third grade fluids were obtained (Mohyuddin, 2005), but the same work have not done for fourth grade fluids, thus there is no clue about regions and signs of fourth grade fluids parameters. Thus in this paper, distinct from other researchers we also consider \( B < 0 \).
APPLIED METHODS

To explain the basic idea of WRMs (Hatami et al., 2013; Necati Özisik, 1993), consider a following differential equation,

\[ D(u) - f(s) = 0, \quad y \in \Omega \]  

(16)

where \( D \) is a general differential operator, \( u \) is an unknown function and \( f(s) \) is a known analytical function. We can assume that the approximate solution of Eq. (16) \( \tilde{u} \) can be written as a linear combination of basic functions chosen from a linearly independent set. That is

\[ u \approx \tilde{u} = \sum_{i=1}^{n} C_i \phi_i \]  

(17)

which satisfies the essential boundary conditions of the problem. Now, substitution of Eq. (17) into Eq. (16) yields:

\[ R(s) = D(\tilde{u}) - f(s) \neq 0 \]  

(18)

where \( R \) is an error or residual since \( \tilde{u} \) is not the exact solution of Eq. (16). Now the problem of finding an approximate solution becomes one of adjusting the values of \( C_i \) so that the residual stays close to zero in some average sense over the domain. That is:

\[ \int_{\Omega} R(s) W_i(s) ds = 0, \quad i = 1, 2, ..., n, \]  

(19)

where \( W_i \) is a weight function. To make a system of \( n \) algebraic equations for the determination of \( n \) \( C_i \) coefficients, the number of weight functions \( W_i \) must be exactly equal to the number of unknown constants \( C_i \). Different methods have been proposed for determination of these unknown coefficients. Collocation (CM), Galerkin (GM) and Least Square (LSM) methods were briefly introduced below.

COLLOCATION METHOD

In Collocation (Necati Özisik, 1993), to make a system of \( n \) algebraic equations for the determination of \( n \) \( C_i \) coefficients, in \( n \) different locations, the residual \( R \) is forced to zero,

\[ R(s_i) = 0, \quad s_i \in \Omega, \quad i = 1, 2, ..., n, \]  

(20)

which can be obtained by considering the weight functions as follows:

\[ W_i(s) = \delta(s - s_i) \]  

(21)

where \( \delta \) is a Dirac delta function,

\[ \delta(s - s_i) = \begin{cases} 1 & s = s_i \\ 0 & \text{otherwise} \end{cases} \]  

(22)

The basic assumption is that the residual does not deviate much from zero between the collocation locations.

GALERKIN METHOD

In Galerkin (Necati Özisik, 1993), the derivative of the approximate function with respect to the unknown constants \( C_i \) is the weight functions, that is:

\[ W_i = \frac{\partial \tilde{u}}{\partial C_i}, \quad i = 1, 2, ..., n. \]  

(23)

LEAST SQUARE METHOD

In Least Square (Necati Özisik, 1993), the integral of the square of residual \( R \),

\[ S = \int_{\Omega} R^2(s) ds, \]  

(24)

is minimized over the domain \( \Omega \). That is,

\[ \frac{\partial S}{\partial C_i} = 2 \int_{\Omega} R(s) \frac{\partial R}{\partial C_i} ds = 0. \]  

(25)

Have Eq. (19) in mind, it seems that the weight functions here are

\[ W_i = 2 \frac{\partial R}{\partial C_i}. \]  

(26)

Since the “2” coefficient can cancel out in the equation, thus the weight functions for the LSM can be considered...
\[ W_i = \frac{\partial R}{\partial C_i} \]  

(27)

**RESULTS AND DISCUSSION**

In this paper, velocity distribution of the thin film flow of a fourth grade fluid on an inclined plane was obtained by CM, GM, LSM and NUM. To solve Eq. (11), a trial function must satisfy the boundary conditions in Eqs. (12-15). On the one hand, each statement in \( \tilde{U}(\eta) \), should contain \( \eta \) to satisfy boundary condition (12). On the other hand, all statements except the last statement should contain \( k(\eta - \eta^k) \) term where \( k \) is an integer to satisfy boundary conditions (12-15). By applying

\[ \tilde{U}(\eta) = C_1(2\eta - \eta^2) + C_2(4\eta - \eta^4) + C_3(6\eta - \eta^6) + \left( \frac{dU}{d\eta} \right)_{\eta=1} \eta \]  

(28)

into Eq. (11), the residual function will be found. Finally, to determine \( C_1, C_2 \) and \( C_3 \), Collocation (CM), Galerkin (GM) and Least Square (LSM) methods were used. To find \( C_1, C_2 \) and \( C_3 \) in CM, the points at which the residual \( R \) is forced to zero were considered in the interval \( 0 \leq \eta \leq 1 \). In GM, by using Eq. (23) the weight functions will be obtained as:

\[ W_1 = (2\eta - \eta^2), \quad W_2 = (4\eta - \eta^4), \quad W_3 = (6\eta - \eta^6). \]  

(30)

In LSM, Eq. (27) can be applied for the weight functions. Table 1, as an example shows \( C_1, C_2 \) and \( C_3 \) defined by CM, GM and LSM for different boundary conditions (12-15) at \( B = -0.03 \) and \( B = 1 \).

Now, by introducing these coefficient into Eq. (28), we have the approximate solution for the problem (11-15). In the following, the solutions of \( U(\eta) \) obtained from data of Table 1 were compared graphically (Figs. 2-5).

| Table 1. \( C_1, C_2 \) and \( C_3 \) values obtained by CM, GM and LSM at \( B = -0.03 \) and 1. |
|---|---|---|---|---|
| B | \( B < 0, B = -0.03 \) | \( B > 0, B = 1 \) |
| Boundary conditions | \( \frac{dU}{d\eta} = \frac{1}{\sqrt{2|B|}} \) | \( \frac{dU}{d\eta} = \frac{1}{\sqrt{2|B|}} \) | \( \frac{dU}{d\eta} = 0 \) | \( \frac{dU}{d\eta} = 0 \) |
| CM | \( C_1 \) | -0.1923733258 | -0.3900786659 | -0.5674385752 | 0.1788355626 |
| | \( C_2 \) | -0.0144962266 | 0.05585693462 | 0.4561954458 | 0.05232710341 |
| | \( C_3 \) | 0.0021677802 | -0.0139874427 | -0.7326619119 | 0.004730132306 |
| GM | \( C_1 \) | -0.1936371655 | -2.278392642 | -1.791738629 | 0.1738422644 |
| | \( C_2 \) | -0.0137165912 | 5.092769440 | 3.50235812 | 0.06275172092 |
| | \( C_3 \) | 0.0019394771 | -4.141288793 | -2.549213560 | -0.001325097493 |
| LSM | \( C_1 \) | -0.1923733258 | -0.3900786659 | 0.5868049419 | 0.1712031880 |
| | \( C_2 \) | -0.0144962266 | 0.05585693462 | -0.0456957594 | 0.06604352896 |
| | \( C_3 \) | 0.0021677802 | -0.0139874427 | 0.0135680148 | -0.002831015368 |

Figure 2. The velocity profile obtained from CM, GM, LSM and NUM for the boundary condition \( U'(1) < 0 \) at \( B = -0.03 \).
Figs. 2 to 4 show the dimensionless velocity profiles, $U(\eta)$ at $B = -0.03$ considering $U'(1) < 0$, $U'(1) > 0$ and $U'(1) = 0$ respectively. Fig. 2 shows that for boundary condition $U'(1) < 0$ at $B = -0.03$ the results generated from CM, GM and LSM were in acceptable agreement with numerical solution but from the physical point of view, it is not the dimensionless velocity profile for fourth grade fluid because $U(\eta)$ is negative and it means there will be an upward fluid flow on inclined plane which is not physically acceptable.

From Figs. 2–5, we can conclude that in this problem the results generated from LSM as compared to CM and GM are in better agreement with the numerical ones for different values of non-Newtonian parameters and constants. Thus by utilizing LSM the effect of non-Newtonian parameter $B$ on the dimensionless velocity profile has been illustrated in Fig. 6 and Fig. 7 for boundary conditions $U'(1) = 0$ and $U'(1) > 0$ respectively. Since the results for boundary condition $U'(1) < 0$ from the physical viewpoint was incorrect so
it has not considered here. It is observed in Fig. 6 that increases in values of parameter $B$, decreases the dimensionless velocity profile and in Fig. 7 that increases in values of parameter $B$, increases the dimensionless velocity profile.

**CONCLUSIONS**

In this work, the steady thin film flow of an incompressible fourth grade fluid on an inclined plane was analyzed using CM, GM, LSM. The results generated from CM, GM and LSM are compared employing a numerical method for validation. Three different boundary conditions on the free surface of the thin film fluid were considered. Based on the obtained graphical results the main conclusions of the study can be summarized as follows:

- The approximate results of LSM as compare to CM and GM for all Boundary conditions and greater regions of dimensionless non-Newtonain parameter $B$ were found to be in acceptable agreement with numerical solutions.

- For boundary condition $U'(1) < 0$, the results was physically incorrect.

- For boundary condition $U'(1) = 0$, while increasing the value of parameter $B$, the dimensionless velocity profile decreases.

- For boundary condition $U'(1) > 0$, while increasing the value of parameter $B$, the dimensionless velocity profile increases.

**REFERENCES**


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