A RESTRICTION ON CENTRALIZERS IN FINITE NON-ABELIAN GROUPS

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ABSTRACT

For a given group $G$, we consider the finite non-abelian groups $G$ for which $|C_G(g):<g>| \leq m$ for every $g \in G \setminus Z(G)$. We show that the order of $G$ can be bounded in terms of $m$ and the largest prime divisor of the order of $G$. Our approach relies on dealing first with the case where $G$ is a non-abelian finite $p$-groups. In the situation, if we take $|C_G(g):<g>| \leq m$ for every $g \in G \setminus Z(G)$, we show that $|C_G(g):<g>| \leq m$ with the only exception of $G_{ab}$. We denote by $\Omega_i(G)$ the subgroup generated by the elements of $G$ of order at most $p^i$, and $G^{p^i}$ is the subgroup generated by the $p^i$th powers of all elements of $G$.

Given a non-abelian groups $G$, it makes sense to impose restrictions on the centralizers of non-central element, and ask what the effect is on the whole of $G$. For example, we may ask what happens if we require that $|C_G(g):<g>| \leq m$ for every $g \in G \setminus Z(G)$, if we take into account that every element commutes with itself, and put a bound on $|C_G(g):<g>|$ as $g$ runs over $Z(G)$. Let us define the maximum centralizer index of a non-abelian finite groups $G$ as

$$mc(G) = \max\{|C_G(g):<g>| \mid g \in G \setminus Z(G)\}.$$ 

then the goal of this paper is to get bounds for the order of $G$ under the condition that $mc(G) = 1$ and $mc(G) = p^k$. If $G$ is non-abelian of order $pq$, with $p$ and $q$ primes, that $m = 1$ but $|G|$ is unbounded.

Proof

If $G \cong Q_8$ or $G$ is non-abelian of order $pq$ with $p$ and $q$ primes, then it is clear that $mc(G) = 1$.

$$Q_8 = \{+1, -1, i, -i, j, -j, k, -k\} \Rightarrow |Q_8| = 8$$

and

$$<i> = \{+1, -1, i, -i\} \Rightarrow |<i>| = 4$$

we know that

$$1 \times i = i \times 1 \Rightarrow i = i$$

$$i \times i = i \times i \Rightarrow -1 = -1$$

$$(-i) \times i = i \times (-i) \Rightarrow 1 = 1$$

$$(-1) \times i = i \times (-1) \Rightarrow -i = -i$$

So

$$C_{Q_8}(i) = \{+1, -1, i, -i\} \Rightarrow |C_{Q_8}(i)| = 4$$

KEYWORDS: non-abelian finite $p$-groups, $g \in G \setminus Z(G)$.

Remark:

If $G$ is a finite $p$-group then $\Omega_i(G)$ denotes the subgroup generated by the elements of $G$ of order at most $p^i$, and $G^{p^i}$ is the subgroup generated by the $p^i$th powers of all elements of $G$.

Main Results

This section is devoted to obtaining a bound for the order of a non-abelian finite $p$-group $G$ given that $mc(G) = 1$. Actually, we will get the best possible bound in terms of $p^{d+1}$. Let $G$ be a finite non-abelian group. Then $C_G(g) = <g>$ for every $g \in G \setminus Z(G)$ if and only if $G \cong Q_8$ or $G$ is non-abelian of order $pq$.

Lemma

If $G \cong Q_8$ or $G$ is non-abelian of order $pq$ with $p$ and $q$ primes, then it is clear that $mc(G) = 1$.

$$Q_8 = \{+1, -1, i, -i, j, -j, k, -k\} \Rightarrow |Q_8| = 8$$

and

$$<i> = \{+1, -1, i, -i\} \Rightarrow |<i>| = 4$$

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So

$$C_{Q_8}(i) = \{+1, -1, i, -i\} \Rightarrow |C_{Q_8}(i)| = 4$$

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and

$$|C_{Q_{0}}(i) : < i > | = \frac{|C_{Q_{0}}(0)|}{|< i > |} = 1$$

hence

$$mci(Q_{0}) = \max \{ |C_{Q_{0}}(i)| : i \in Q_{0}\backslash Z(Q_{0}) \} = 1$$

let now $G$ be a group such that $mci(G) = 1$, and let $P$ be a Sylow $p$-subgroup of $G$ which is not central in $G$. If $A$ is maximal abelian subgroup of $P$, then $A \not\leq Z(G)$ because $A \leq P \not\leq Z(G)$ and $C_{G}(g) : < g >$ for every $g \in A \backslash Z(G)$. Thus we get the following:

1) $A$ is cyclic and $|A : A \cap Z(G)| = p$.

2) If $p \neq q$ is prime, then $Q \cap C_{G}(P) = 1$ for every Sylow $q$-subgroup $Q$ of $G$. In particular, $Q \cap Z(G) = 1$.

If follows from (1) and [8], that $P$ is either cyclic or isomorphic to $Q_{0}$. In particular, $G$ is divisible by at least two primes. Let $K$ be an arbitrary non-trivial Sylow subgroup of $G$.

**Definition**

Let $G$ be a finite $p$-group. We say that $G$ is $p$-central if $p > 2$ and $\Omega_{2}(G) \leq Z(G)$, or if $p = 2$ and $\Omega_{2}(G) \leq Z(G)$.

The $p$-central $p$-groups are somehow dual to powerful $p$-groups, which are defined by the condition $G' \leq G^{p}$ if $G$ is odd, or $G' \leq G^{p} = 2$.

A well-known property of powerful $p$-groups is that $[G^{\pi^{i}}, G^{\pi^{i+2}}] \leq [G^{\pi^{i}}, G^{\pi^{i+3}}]$ for every $i \geq 0$.

**Lemma 2.3**

Let be a $p$-central $p$-group, then

$$|\Omega_{i+2}(G) : \Omega_{i+1}(G)| \leq |\Omega_{i+1}(G) : \Omega_{i}(G)|$$

for every $i \geq 0$.

**Proof:**

**Theorem**

Let $G$ be a non-abelian finite $p$-group such that $mci(G) = p^{k}$. Then $|G| \leq p^{2k+2}$, unless $G \cong Q_{0}$.

**Proof:**

Assume first that $p$-central, and let $r$ be such that $\Omega_{r}(G) \leq Z(G)$ but $\Omega_{r+1}(G) \not\leq Z(G)$.

If $|\Omega_{r+1}(G) : \Omega_{r}(G)| = p$ then

By taking (2) into account. For every Sylow $q$-subgroup $K \cap C_{G}(P) = 1$ and $K \leq Z(G) = 1$. So $K$ is not central in $G$, and so $K$ can play the role of $P$ in the previous paragraph. Hence $A : A \cap Z(G)$ is prime for every maximal abelian subgroup $A$ of $K$. But by (2) above (corresponding to $P$) we know that $K \cap Z(G) = 1$. It follows that every maximal abelian subgroup of $K$ is of prime order, and so $K$ is of prime order.

Hence the order of $G$ is square-free. By taking [7] into account, $G$ is the semidirect product of two cyclic subgroups of coprime orders. According to (2), these two cyclic subgroups must be of prime order. We conclude that $G$ is a non-abelian group of order $pq$ for two primes $p$ and $q$ as desired.

By Theorem B of [2], we know that $G/\Omega_{2}(G)$ is $p$-central, and that $\exp(\Omega_{1}(G)) \leq p^{i}$ for every $i \geq 1$. Then

$$|\Omega_{i+2}(G) : \Omega_{i+1}(G)| = |\Omega_{i+2}(G) : \Omega_{i+1}(G)|$$

and if we work with $G/\Omega_{2}(G)$ instead of $G$, it suffices to prove that $|\Omega_{2}(G) : \Omega_{1}(G)| \leq |\Omega_{1}(G)|$.

This follows immediately if we see that the map $x \mapsto x^{p}$ is an automorphism from $\Omega_{2}(G)$ to $\Omega_{2}(G)$. This result is obvious if $p = 2$, since $\Omega_{2}(G)$ is then abelian. If $p > 2$ then $\Omega_{2}(G) \leq Z_{2}(G)$, since $G/\Omega_{2}(G)$ is $p$-central. Hence $\Omega_{2}(G)$ has class at most 2, and $\exp(\Omega_{2}(G)) \leq \exp(\Omega_{1}(G)) \leq p$. Thus $(xy)^{p}$ for every $x, y \in \Omega_{2}(G)$, and we are done.
\[ |\Omega_{i+1}(\frac{G}{\Omega_i(G)}) : \Omega_i(\frac{G}{\Omega_i(G)})| = |\Omega_{i+r+1}(G) : \Omega_{i+r}(G)| \]
\[ \leq |\Omega_{i+r}(G) : \Omega_{i+r+(i-1)}(G)| \]
\[ \leq |\Omega_{i+r+(i-1)}(G) : \Omega_{i+r+(i-2)}(G)| \]
\[ \leq \cdots \leq |\Omega_{i+r+1}(G) : \Omega_r(G)| = p \]

for every \( i \geq 0 \), by lemma 2.3. It follows that \( \frac{G}{\Omega_i(G)} \) is cyclic, and then \( G \) is abelian.

since \( \Omega_r(G) \leq Z(G) \). Thus we have
\[ |\Omega_{r+1}(G) : \Omega_r(G)| \geq p^2 \quad (2.1) \]

Again by lemma 2.3, it follows that
\[ |\Omega_r(G)| = \frac{|\Omega_1(G)|}{|\Omega_0(G)|} \cdot \frac{|\Omega_2(G)|}{|\Omega_1(G)|} \cdots \frac{|\Omega_{r-1}(G)|}{|\Omega_{r-2}(G)|} \cdot \frac{|\Omega_r(G)|}{|\Omega_{r-1}(G)|} \]
\[ = |\Omega_1(G) : \Omega_0(G)| \cdot |\Omega_2(G) : \Omega_1(G)| \cdots |\Omega_r(G) : \Omega_{r-1}(G)| \]
\[ \geq |\Omega_1(G)| \cdot p^2 \cdot \cdots \cdot p^2 = p^{2r-2}|\Omega_1(G)| \]

and \( |\Omega_0(G) = 1| \), then
\[ |\Omega_2(G) : \Omega_0(G)| |\Omega_2(G) : \Omega_1(G)| \cdots |\Omega_r(G) : \Omega_{r-1}(G)| = \prod_{i=0}^{r-1} |\Omega_{i+1}(G) : \Omega_i(G)| \]
so
\[ |\Omega_r(G)| = \prod_{i=0}^{r-1} |\Omega_{i+1}(G) : \Omega_i(G)| \geq p^{2r-2}|\Omega_1(G)| \]

since \( |G : G^p| \leq |\Omega_1(G)| \) so \( \frac{|G|}{|G^p|} \leq |\Omega_1(G)| \) by theorem C of [2], we get
\[ \frac{|G|}{|G^p|} \leq |\Omega_1(G)| \leq \frac{|\Omega_2(G)|}{|\Omega_1(G)|} = |G| \leq \frac{|\Omega_2(G)|}{p^{2r-2}} \quad (2.2) \]

Let us choose an arbitrary element \( g \in \Omega_{r+1}(G) \setminus Z(G) \) since \( \Omega_{r+1}(G) \leq Z_2(G) \) we have
\[ |\Omega_{r+1}(G), G^p| = |\Omega_{r+1}(G)^p, G| \leq |\Omega_r(G), G| = 1. \]

So \( G^p \leq C_0(g) \), and hence
\[ mci(G) = \max(|C_0(g) : < g >| \in \Omega_{r+1}(G) \setminus Z(G)) = p^k \]
we have
\[ \max|C_0(g) : < g >| \geq |C_0(g) : < g >| \geq |G^p : < g >| \geq |G^p| \geq \frac{|G^p|}{p^{r+1}} \]

hence
\[ p^k \geq |C_0(g) : < g >| \geq |G^p : < g >| \geq \frac{|G^p|}{p^{r+1}} \]

and
\[ |G^p| \leq p^{k+r+1} \quad (2.3) \]

Similarly
\[ p^k \geq |C_G(g) : < g >| \geq |G^p \Omega_r(G) < g > : \Omega_r(G) < g > \} > |\Omega_r(G) < g > | \]
\[ = |G^p \Omega_r(G) < g > : \Omega_r(G) < g > | / |\Omega_r(G)| / p^{r(2.4)} \]
and in particular \(|G^p \Omega_r(G) < g > : \Omega_r(G) < g > | = 1\), we have
\[ p^k \geq \frac{|\Omega_r(G)|}{p^r} \Rightarrow |\Omega_r(G)| \leq p^{k+r}. \quad (2.5) \]

Now we consider separately the cases \(G^p \not\leq \Omega_r(G)\) and \(G^p \leq \Omega_r(G)\). Assume that \(G^p \not\leq \Omega_r(G)\). Since \(\frac{|\Omega_r(G)|}{|\Omega_r(G)|} \) is an elementary abelian \(p\)-group of order at least \(p^2\) and since \(\frac{Z(G)}{\Omega_r(G)}\) is a proper subgroup of \(\frac{|\Omega_r(G)|}{|\Omega_r(G)|} \)

\[ \bigcap_{g \in \Omega_r(G) \setminus Z(G)} \Omega_r(G) < g > : \Omega_r(G) < g > \]
Consequently, we can choose \(g \in \Omega_r(G) \setminus Z(G)\) such that \(G^p \not\leq g > : \Omega_r(G)\). Since \(G^p \not\leq g > : \Omega_r(G)\), so \(G^p \Omega_r(G) < g > : \Omega_r(G) < g > | = p^r\) and by taking (2.4) into account, we can improve (2.5) to
\[ |\Omega_r(G)| \leq p^{k+r-1} \quad (2.6) \]

Also, if \(G^p \leq \Omega_r(G)\) then \(|G^p| \leq |\Omega_r(G)| \leq p^{r+k}\) by (2.5) we can improve (2.3) to
\[ |G^p| \leq p^{r+k} \quad (2.7) \]

Thus we can combine either (2.3) and (2.6) and then use (2.2)
\[ |G| \leq \frac{|G^p||\Omega_r(G)|}{p^{2r-2}} \leq \frac{p^{k+r+1}p^{k+r+1}}{p^{2r-2}} = p^{2k+2} \]
or (2.5) and (2.7) and then use (2.2)
\[ |G| \leq \frac{|G^p||\Omega_r(G)|}{p^{2r-2}} \leq \frac{p^{k+r}p^{k+r}}{p^{2r-2}} = p^{2k+2} \]
to get \(|G| \leq p^{2k+2}\) in any case. This completes the proof when \(G\) is \(p\)-central.

Assume now that \(G\) is not \(p\)-central, and suppose that \(|G| > p^{2k+2}\). We are going to prove that \(G \cong Q_8\). Put \(e = 0\) or 1 according as \(p > 2\) or \(p = 2\). Let us choose a subgroup \(A\) of \(G\) which is maximal in the set of abelian normal subgroups of \(G\) of exponent at most \(p^{1+e}\).

Also \(\Omega_{1+e}(C_G(A)) = A\). If \(A \not\leq Z(G)\) then we get \(\Omega_{1+e}(G) \leq Z(G)\), which is not the case.

Thus \((A \cap Z(G)) \setminus (A \cap Z(G))\) is not empty. Let \(t \in (A \cap Z(G)) \setminus (A \cap Z(G))\) Then
\[ p^k \geq |C_G(t) : < t >| \geq |C_G(t)| / p^{1+e} \quad (2.8) \]

since \(t \in A\) and subgroup \(A\) of \(G\) which is maximal in the set of abelian normal subgroups of \(G\) of exponent at most \(p^{1+e}\), so \(|< t >| = p^{1+e}\). In particular case
\[ |A| \leq |C_G(t)| \leq p^{k+1+e} \quad (2.9) \]
thus \(A \cap Z(G) \not\subseteq A\) then \(|A \cap Z(G)| \leq |A| \leq p^{k+1+e}\), we have
\[ |A \cap Z(G)| \leq |A| / p \leq p^{k+e} \quad (2.10) \]
On the other hand \(G' = \{[t, x]|x \in G\}\), also \(\{[t, x]|x \in G\} \subseteq A \cap Z(G)\), so
\[ |G : C_G(t)| = |\{[t, x]|x \in G\}| \leq |A \cap Z(G)| \quad (2.11) \]
Consequently of (2.9), (2.10), and (2.11) relations, we have
\[ |G| = |G : C_G(t)| |C_G(t)| \leq |A \cap Z(G)| |C_G(t)| \]

\[ |G : C_G(t)| = |\{[t, x]|x \in G\}| \leq |A \cap Z(G)| \quad (2.11) \]
Consequently of (2.9), (2.10), and (2.11) relations, we have
\[ |G| = |G : C_G(t)| |C_G(t)| \leq |A \cap Z(G)| |C_G(t)| \]
\[ \leq p^{k+\varepsilon} \times p^{k+1+2\varepsilon} = p^{2k+1+2\varepsilon} \]
\[ \Rightarrow |G| < p^{2k+1+2\varepsilon} \]

Since \(|G| > p^{2k+2}\), this implies that \(\varepsilon = 0\) according as \(p = 2\) and \(|G| = 2^{2k+3}\). Thus all inequalities (2.8), (2.9), (2.10), and (2.11) are equalities. So \(C_G(t) = A\) by (2.9) relation and consequently \(Z(G) \leq A\), that

\[ A \cap Z(G) = Z(G) \Rightarrow |A \cap Z(G)| = |Z(G)| = p^{k+\varepsilon} = 2^{k+1} \]
\[ \Rightarrow |A \cap Z(G)| = 2^{k+1} \]

and

\[ |A| = p^{k+1+\varepsilon} = p^{k+2} = 2^{k+2} \]

hence

\[ [A: Z(G)] = \frac{|A|}{|Z(G)|} = \frac{2^{k+2}}{2^{k+1}} = 2 \Rightarrow [A: Z(G)] = 2 \]

and

\[ Z(G) = \{[t, x] | x \in G\} \]

since \(|A: Z(G)| = 2\), we have \(A \leq Z_2(G)\). So any element of \(A \setminus Z(G)\) is a valid choice for \(t\). Also

\[ [A: G^2] = [A^2, G] \leq [Z(G), G] = 1, \]

and so \(G^2 \leq C_G(t) = A\). If \(g^2 \in A \setminus Z(G)\) for some \(g \in G\) then we can choose \(t = g^2\), and \(g \in C_G(t) \setminus A\), which is a contradiction. We conclude that \(G^2 \leq Z(G)\). Since \(G' \leq G^2\) so \(G\) is a group of class 2, and \(G' = \{[t, x] | x \in G\} = Z(G)\), for every \(t \in A \setminus Z(G)\) \(2.12\)

In particular \(|G'| = |Z(G)| = p^{k+1} = 2^{k+1}\). Also

\[ \exp Z(G) = \exp G' = \exp G / Z(G) = 2 \]

by using that \(G\) is of class 2. Hence

\[ \exp G / Z(G) = \exp G / \exp Z(G) = 2 \Rightarrow \exp G = 4 \]

Thus if we choose an arbitrary element \(g \in G \setminus G'\), then \(< g > Z(G)\) is a normal abelian subgroup of \(G\) of exponent at most 4. By embedding this subgroup in a maximal abelian normal subgroup of exponent at most 4, we see that \(g\) can play the same role as \(t\) above, and in particular

\[ G' = \{[g, x] | x \in G\} \]

by (2.12). Also \(|G: G'| \geq |G| \). Thus

\[ 2^{2k+3} = |G| = |G: G'| |G'| \leq |G'|^2, |G'| = |G|^3 = (2^{k+1})^3 = 2^{2k+3} \quad (2.13) \]

and \(k = 0\), hence \(mci(G) = 1\), and \(G \cong Q_8\) by lemma 2.1.

Alternatively, we get \(|G| = 8\) from (2.13), and so \(G \cong Q_8\) or \(G \cong D_8\).

Let \(n > 2\), \(D_{2n} = \{x^i y^j | i = 0, 1, j = 0, 1, ... , n - 1\}\) be an non-abelian group.

Then \(x^i y^j = x^\alpha y^\beta \Leftrightarrow \begin{cases} i \equiv 2 \alpha \\ j \equiv n \beta \end{cases}\) and \(|D_{2n}| = 2n\). Therefore

\[ D_8 = \{e, y, y^2, y^3, x, xy, xy^2, xy^3\} \Rightarrow |G| = |D_8| = 8 \]

and

\[ < xy \geq \{e, xy\} \Rightarrow |< xy | = 2 \]
we know that
\[ e \times xy = xy \times e \Rightarrow xy = xy \]
\[ xy \times y^2 = y^2 \times xy \Rightarrow xy^3 = xy^3 \]
\[ xy \times xy = xy \times xy \Rightarrow y^2 = y^2 \]
\[ xy \times xy = xy \times xy^3 \Rightarrow e = e. \]

So
\[ C_G(xy) = \{ e, xy, y^2, xy^3 \} \Rightarrow |C_G(xy)| = 4 \]
and
\[ |C_G(xy) : < xy > | = \frac{|C_G(xy)|}{|< xy >|} = \frac{4}{2} = 2 \]
hence
\[ mci(D_a) = \max \{|C_G(xy) : < xy >| \mid xy \in D_a \setminus Z(G) \} \Rightarrow mci(D_a) = 2 \]

Now since \( mci(Q_a) = 1 \) but \( mci(D_a) = 2 \), we necessarily have \( G \cong Q_a \).

Now we present an example which shows that the bound \( |G| \leq p^{2k+2} \) in Theorem (2.4) is best possible.

**Example 2.5**

Let \( p \) be an arbitrary prime, and let \( G \) be the group given by the following presentation:

\[ G = \langle a, b \mid a^{p^{k+1}} = b^{p^{k+1}} = 1, a^b = a^{1+p^k} \rangle. \]

Then \(|G| = p^{2k+2}, Z(G) = \langle a^p, b^p \rangle \) and \( o(g) = p^{k+1} \) for every \( g \in G \setminus Z(G) \).

\[ C_G(g) = \langle a, b \mid a^{p^k} = b^{p^{k+1}} = 1 \rangle \Rightarrow |C_G(g)| = p^{2k+1} \]

And
\[ |C_G(g) : < g > | = \frac{|C_G(g)|}{|< g >|} = \frac{p^{2k+1}}{p^{k+1}} = p^k \]

Therefore
\[ mci(G) = \max \{|C_G(g) : < g >| \mid g \in G \setminus Z(G) \} = p^k \]

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