ANALYTIC SOLUTION OF HYDRODYNAMIC AND THERMAL BOUNDARY LAYERS OVER A FLAT PLATE IN A UNIFORM STREAM OF FLUID WITH CONVECTIVE SURFACE BOUNDARY CONDITION

MORTEZA ABBASI\textsuperscript{1,a}, GHODRATOLLAH HAMZEH NAVA\textsuperscript{b}, IMAN RAHIMI PETROUDI\textsuperscript{c}

\textsuperscript{1}Department of Mechanical Engineering, Sari Branch, Islamic Azad University, Sari, Iran
\textsuperscript{2}School of Mechanical Engineering, College of Engineering, University of Tehran, Tehran, Iran
\textsuperscript{3}Young researchers club, sari branch, Islamic Azad University, sari, Iran

ABSTRACT

An analysis has been performed to study the classical problem of hydrodynamic and thermal boundary layers over a flat plate in a uniform stream of fluid. The governing boundary layer and temperature equations for this problem are reduced to an ordinary form and are solved by Homotopy Analysis Method (HAM), and Numerical solution as Boundary Value Problem (BVP). Velocity and temperature profiles are shown graphically. In this works, the HAM has been used to solve nonlinear differential equations with convective surface boundary condition. The obtained solutions, in comparison with the numeric solutions admit a remarkable accuracy.

KEYWORDS: Homotopy analysis method; Natural convection; convective surface boundary condition

Heat transfer by natural convection frequently occurs in many physical problems and engineering applications such as geothermal systems, heat exchangers, chemical catalytic reactors, fiber and granular insulation, packed beds, petroleum reservoirs and nuclear waste repositories. In review of its importance, the flow of Newtonian and non-Newtonian fluids through two infinite parallel vertical plates has been investigated by numerous authors. The natural convection problem has been carried out by different authors (Bruce and Na., 1967; Shenoy., 1982; Rajagopal and Na., 1985).

Most of problems and scientific phenomena such as heat transfer are inherently of nonlinearity. We know that except a limited number of these problems, most of them do not have exact solutions. Therefore, these nonlinear equations should be solved approximately either numerically or analytically. In the numerical method, stability and convergence should be considered so as to avoid divergence or inappropriate results. Time consuming is another problem of numerical techniques. Analytical solutions often fit under classical perturbation methods (Aude et al., 1998; Fillo and Geer, 1996a). However, as with other analytical techniques, certain limitations restrict the wide application of perturbation methods, most important of which is the dependence of these methods on the existence of a small parameter in the equation. Disappointingly, the majority of nonlinear problems have no small parameter at all.

Recently, several new techniques have been presented to overcome the mentioned difficulties. Some of these techniques include Variational Iteration Method (VIM) (He, 1999a, 2007; He and Wu, 2007; Ganji et al., 2007; Ganji and Sadighi, 2007), decomposition method (Adomian, 1983, 1986), Homotopy Perturbation Method (HPM) (He, 1999b, 2003,2006b; Ganji and Sadighi, 2006; Hosein Nia et al., 2008) and Homotopy Analysis Method (Liao., 2003, 2004, 2012; Ghasempour et al., 2009; Sohouli et al., 2010; Sajid and hayat, 2009; Fooladi et al., 2009; Kimiaeifar et al., 2009b) etc. The Homotopy Analysis method (HAM) is one of the well-known methods to solve the nonlinear equations that does not need to small parameter. This method has been introduced by Liao in 1992. The method has been used in a wide variety of scientific and engineering applications to solve different types of governing differential equations: linear and nonlinear, homogeneous and non-homogeneous, and coupled and decoupled as well. This method offers highly accurate successive approximations of the solution.

In the present work, the equation of the laminar thermal boundary layer over a flat plate with a convective surface boundary condition is solved through HAM. The convergence of the series solution is also explicitly discussed. Obtaining the analytical solution of the models and comparing with numerical result reveal the capability, effectiveness and convenience of HAM. The governing equations for this problem are solved by HAM, and Numerical solution. The numerical solution is performed using the algebra package Maple 14.0, to solve the present case. The software uses a second-order difference scheme combined with an order bootstrap technique with mesh-refinement strategies: the difference scheme is based on either the trapezoid or midpoint rules; the order improvement/accuracy enhancement is either Richardson extrapolation or a method of deferred correction (Aziz., 2006). This method gives successive approximations of high accuracy solution.

PROBLEM STATEMENT AND MATHEMATICAL FORMULATION

Consider the two-dimensional of hydrodynamic and thermal boundary layer flow over a flat plate in a stream of cold fluid at temperature $T_{\infty}$ moving over the top surface of the plate with a uniform velocity $U_{\infty}$ as illustrated in Fig.1 (Hatami et al.2013).
Figure 1. Schematic of Velocity Boundary Layer and Temperature Boundary Layer on a horizontal plate

For the steady two-dimensional flow, the equations of continuity, momentum, and energy Equations are:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \]  

(1)

\[ \frac{D \rho}{Dt} = -\nabla P + \rho \nabla^2 \mathbf{V} + dF, \]  

(2)

\[ \frac{D c_p T}{Dt} = \nabla \cdot (k \nabla T) + \mu \Phi + \dot{q}, \]  

(3)

In above equations \( \rho \) is the fluid density, \( \mu \) is the dynamic viscosity, \( k \) is the thermal conductivity, \( c_p \) is the specific heat at constant pressure, \( t \) denotes time and \( \nabla \mathbf{V}, \nabla T \) represents the fluid velocity and temperature gradient tensor. Using an order of magnitude analysis, the well-known governing Navier–Stokes equations of viscous fluid flow can be greatly simplified within the boundary layer. Using the following boundary layer approximations (Oosthuizen and Naylor, 1999)

\[ u = \frac{u}{u_1} O(1), \quad v = \frac{v}{u_1} O \left( \frac{\delta}{L} \right), \]

\[ x = \frac{x}{L} O(1), \quad y = \frac{y}{\delta} O(1), \]

\[ P = \frac{P}{\rho u_1^2} O(1), \quad \theta = \frac{T - T_\infty}{T_w - T_\infty} O(1), \]  

(4)

The continuity, momentum, and energy Equations describing the flow can be written as:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \]  

(5)

\[ \frac{u}{\partial x} + \frac{\partial T}{\partial y} = \nu \frac{\partial^2 T}{\partial y^2}, \]  

(6)

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}, \]  

(7)

Here \( x \) and \( y \) are, respectively, the directions along and perpendicular to the plate, and \( u \) and \( v \) are the velocity components along \( x \) and \( y \) directions. \( \nu \) is the kinematic viscosity of the fluid, and \( \alpha \) is the thermal diffusivity of the fluid. The appropriate velocity boundary conditions become:

\[ u(x, 0) = v(x, 0) = 0, \]

(8)

\[ u(x, \infty) = U_\infty \]  

(9)

The bottom side of the plate is in contact with a hot fluid at temperature \( T_f \) which provides a heat transfer coefficient \( h_f \).

The thermal boundary conditions may be written as (Aziz., 2009):

\[ k \frac{\partial T}{\partial y}(x, 0) = h_f \left[ T_f - T(x, 0) \right], \]  

(10)

\[ T(x, \infty) = T_\infty. \]  

(11)

To further reduce this Equation into a single ODE, a similarity parameter \( \eta \) is introduced as:

\[ \eta = y \left( \frac{U_\infty}{u} \right)^{\frac{1}{2}} \]  

(12)

Where \( U \) is the withdrawal velocity. Similarly, stream function \( \psi \) can be made dimensionless as:

\[ f(\eta) = \frac{\psi}{U_\infty \sqrt{\frac{u x}{U_\infty}}} \]  

(13)

Where \( f(\eta) \) is the dimensionless stream function. In terms of the new variables, the velocity components \( u \) and \( v \) automatically satisfy Equation (1). The governing Equations (6) and (7) in terms of the new variables \( f \) and \( \theta \) can be written as:

\[ 2 f''' + f f'' = 0, \]  

(14)
\[ \theta'' + \frac{1}{2} \text{Pr} f \theta' = 0, \]
\[ f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \]
\[ \theta'(0) = -a[1 - \theta(0)], \quad \theta(\infty) = 0, \]

Where
\[ a = \frac{h_f}{k} \sqrt{\frac{\nu x}{U_\infty}} \]  

For the energy equation to have a similarity solution, the quantity \( a \) must be a constant and not a function of \( x \) as in Equation. (17). This condition can be met if the heat transfer coefficient \( h_f \) is proportional to \( x^{-1/2} \). We therefore assume
\[ h_f = cx^{-\frac{1}{2}} \]  

Where \( c \) is a constant. With the introduction of Equation. (18) into Equation. (17), we have:
\[ a = \frac{c}{k} \sqrt{\frac{\nu}{U_\infty}} \]  

In the next sections we shall solve the system of Equations. (14) and (15) by using the HAM. The equations are coupled and non-linear.

**IMPLEMENTATION OF THE HOMOTOPY ANALYSIS METHOD**

For HAM solutions, we choose the initial guesses and auxiliary linear operators in the following form:

\[ f_0(\eta) = -1 + \eta + e^{-\eta}, \quad \Theta_0(\eta) = \frac{1}{2} a e^{-\eta}, \]
\[ L_1(f) = f'' + f', \quad L_2(\Theta) = \Theta' + \Theta' \]
\[ L_1(C_1 + C_2 \eta + C_3 e^{-\eta}) = 0, \quad L_2(C_4 + C_5 e^{-\eta}) = 0, \]

And \( c_i \) \((i = 1 - 5)\) are constants. Let \( P \in [0,1] \) denotes the embedding parameter and \( h \) indicates non-zero auxiliary parameters. We then construct the following Equations:

Zeroth –order deformation Equations
\[ (1 - P) L_1 f(\eta; p) - f_0(\eta) = ph_1 H(\eta) N_1 f(\eta; p) \]  
\[ f(0; p) = 0; \quad f'(0; p) = 0; \quad f'(\infty; p) = 1 \]  
\[ (1 - P) L_2 \Theta(\eta; p) - \Theta_0(\eta) = ph_2 H(\eta) N_2 \Theta(\eta; p), f(\eta; p) \]  
\[ \Theta'(0; p) - \Theta(0; p) = -a; \quad \Theta(\infty; p) = 0, \]

\[ N_1 f(\eta; p) = 2 \frac{d^2 f(\eta; p)}{d\eta^2} + f(\eta; p) \frac{d^2 f(\eta; p)}{d\eta^2} = 0, \]  
\[ N_2 f(\eta; p), \Theta(\eta; p) = \frac{d^2 \Theta(\eta; p)}{d\eta^2} + \frac{1}{2} \text{Pr} f(\eta; p) \frac{d\Theta(\eta; p)}{d\eta} = 0, \]

For \( p = 0 \) and \( p = 1 \) we have
\[ f(\eta; 0) = f_0(\eta) \quad f(\eta; 1) = f(\eta) \]
\[ \Theta(\eta; 0) = \Theta_0(\eta) \quad \Theta(\eta; 1) = \Theta(\eta) \]

When \( p \) increases from 0 to 1 then \( f(\eta; p) \) and \( \Theta(\eta; p) \) vary from \( f_0(\eta) \) and \( \Theta_0(\eta) \) to \( f(\eta) \) and \( \Theta(\eta) \). By Taylor's theorem and using Equations (30) and (31), \( f(\eta; p) \) and \( \Theta(\eta; p) \) can be expanded in a power series of \( p \) as follows:
\[ f(\eta; p) = f_0(\eta) + \sum_{n=1}^{\infty} f_n(\eta) p^n, \quad f_n(\eta) = \frac{1}{m!} \frac{\partial^n f(\eta; p)}{\partial p^n} \]  
\[ \Theta(\eta; p) = \Theta_0(\eta) + \sum_{n=1}^{\infty} \Theta_n(\eta) p^n, \quad \Theta_n(\eta) = \frac{1}{m!} \frac{\partial^n \Theta(\eta; p)}{\partial p^n} \]

In which \( h_1 \) and \( h_2 \) is chosen in such a way that these two series are convergent at \( p = 1 \), therefore we have through Equations (32) and (33) that
Figure 2. The $h$ - validity of $f''(0)$ and $\Theta'(0)$ for $Pr = 3$, $a = 0.9$ and 6-8-10-11th-order of approximation

$m$th -order deformation Equations

\begin{align}
L_m[f_m(\eta) - \chi_m f_{m-1}(\eta)] &= h_m H(\eta) R_m'(\eta) \\
f_m(0) &= 0; \quad f_m'(0) = f_m'(\infty) = 0, \\
L_2[\Theta_m(\eta) - \chi_m \Theta_{m-1}(\eta)] &= h_2 H(\eta) R_m^\Theta(\eta) \\
\Theta'_m(0) - \Theta_m(0) &= 0; \quad \Theta_m(\infty) = 0.
\end{align}

\begin{align}
R_m'(\eta) &= 2 f_{m-1}'' + \sum_{n=0}^{m-1} [f_{m-1-n}^* f_n], \\
R_m^\Theta(\eta) &= \Theta_m^* + \frac{1}{2} Pr \sum_{n=0}^{m-1} \Theta_{m-1-n}^* f_n.
\end{align}

Now we determine the convergency of the result, the differential Equation, and the auxiliary function according to the solution expression. So let us assume:

\begin{equation}
H(\eta) = e^{-\eta}
\end{equation}

We have found the answer by maple analytic solution device. Two first deformations of the coupled solutions are presented below.

\begin{align}
f_1(\eta) &= -\frac{7}{12} h_1 e^{-\eta} - \frac{1}{18} h_1 e^{3\eta} + \frac{1}{4} h_1 e^{2\eta}, \\
\Theta_1(\eta) &= -\frac{1}{24} h_2 e^{3\eta} Pr a + \frac{1}{4} h_2 e^{2\eta} a \\
&+ \frac{1}{4} Pr a h_2 \left( - \frac{1}{2} e^{-\eta} - \frac{1}{4} e^{2\eta} \right) \\
&- e^{-\eta} \left( - \frac{11}{96} Pr a h_2 + \frac{3}{8} h_2 a \right)
\end{align}

The solutions $f_{11}(\eta)$ and $\Theta_{11}(\eta)$ were too long to be mentioned here, therefore, they are shown graphically.

Convergence Of The HAM Solution

As pointed out by Liao, the convergence region and rate of solution series can be adjusted and controlled by means of the auxiliary parameter $h$. In general, by means of the so-called $h$-curve, it is straightforward to choose an appropriate range for $h$ which ensures the convergence of the solution series. To influence of $h$ on the convergence of solution, we plot the so-called $h$-curve of $f''(0)$ and $\Theta'(0)$ by 11th-order approximation, as shown in Fig. 2-3.

The solutions converge for $h$ values which are corresponding to the horizontal line segment in $h$ curve. In our case study, for $Pr = 3$ and $a = 0.9$ the ranges for values of $f''(0)$ and $\Theta'(0)$ are $-1 < h_1 < 0$ and $-2 < h_2 < 0$ and for $Pr = 10$ and $a = 0.6$ the ranges for values of $f''(0)$ and $\Theta'(0)$ are $-0.8 < h_1 < 0.1$ and $-2 < h_2 < -0$ respectively.
RESULTS AND DISCUSSION

Figs 4 – 7 show comparison between the numerical solution and HAM solution for \( f'(\eta), \Theta(\eta) \) and \( \Theta'(\eta) \) with different values of Prandtl number. According to Figs 4 – 7 this method provides that the approximations obtained by HAM converge to the exact solution quite fast.
Moreover, Figs 5 to 7 are prepared in order to see the effects of parameter $a$, on the Temperature distributions on a flat plate with convective boundary condition. These Figures shows that, for each Prandtl number, both $\Theta(\eta)$ and $\Theta'(\eta)$ increase as $a$ increases.

![Figure 6. Effects of parameter $a$ on 20th-order approximation of the temperature distribution at Pr=6](image)

Figure 6. Effects of parameter $a$ on 20th-order approximation of the temperature distribution at Pr=6

As long as, the thermal resistance on the hot fluid side is inversely proportional to $h_f$. Thus, $a$ increases, the hot fluid side convection resistance decreases and consequently, the surface temperature $\Theta(\eta)$ should increase as indeed is the case in Figs. 5-7.

![Figure 7. Effects of parameter $a$ on 20th-order approximation of the temperature distribution at Pr=10](image)

Figure 7. Effects of parameter $a$ on 20th-order approximation of the temperature distribution at Pr=10

TRANSFORMATIONS

In the present work we have applied the Homotopy analysis method (HAM) to compute the classical problem of hydrodynamic and thermal boundary layers over a flat plate in a uniform stream of fluid with convective surface boundary condition. Results clearly shows that Homotopy Analysis Method applied to the nonlinear equations was capable of solving them with successive rapidly convergent approximations without any restrictive assumptions or transformations causing changes in the physical properties of the problem. The results show us the validity and great potential of the HAM for Heat Transfer equations in engineering.

CONCLUSIONS

REFERENCES

Aziz A.; 2009. A similarity solution for laminar thermal boundary layer over a flat plate with a convective surface


